1.) **Subgroups**

Let \((G, \vee)\) be a group and let \(H \subset G\) with \(H \neq \emptyset\). Show that \(H\) is a subgroup of \(G\) if and only if \(a, b \in H\) implies \(a \vee b^{-1} \in H\).

(5 points)

2.) **Banach space**

Let \(B\) be a \(K\)-vector space (\(K = \mathbb{R}\) or \(\mathbb{C}\)) with null vector \(\theta\). Let \(\|\ldots\| : B \to \mathbb{R}\) be a mapping such that

(i) \(\|x\| \geq 0 \quad \forall \ x \in B,\ \text{and} \ \|x\| = 0 \iff x = \theta\).

(ii) \(\|x + y\| \leq \|x\| + \|y\| \quad \forall \ x, y \in B\).

(iii) \(\|\lambda x\| = |\lambda| \cdot \|x\| \quad \forall \ x \in B, \lambda \in K\).

Then we call \(\|\ldots\|\) a **norm** on \(B\), and \(\|x\|\) the **norm** of \(x\).

Further define a mapping \(d : B \times B \to \mathbb{R}\) by

\[d(x, y) := \|x - y\| \quad \forall \ x, y \in B\]

Then we call \(d(x, y)\) the **distance** between \(x\) and \(y\).

a) Show that \(d\) is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space \(B\) with distance/metric \(d\) is complete, then we call \(B\) a **Banach space** or **B-space**.

b) Show that \(\mathbb{R}\) and \(\mathbb{C}\), with suitably defined norms, are B-spaces. (For the completeness of \(\mathbb{R}\) you can refer to §4.5 example (3), and you can assume the completeness of \(\mathbb{C}\).)

Now let \(B^*\) be the dual space of \(B\), i.e., the space of linear functionals \(\ell\) on \(B\), and define a norm of \(\ell\) by

\[\|\ell\| := \sup_{\|x\|=1} \{ |\ell(x)| \}\]

c) Show that the such defined norm on \(B^*\) is a norm in the sense of the norm defined on \(B\) above.

**Note:** In case you’re wondering: \(B^*\) is complete, and hence a B-space, but the proof of completeness is difficult.

(8 points)