1.4.8. **Hilbert space**

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of the definition in Problem 1.4.7. (aka the second problem on the 2021 Midterm).

*hint:* Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings $\ell$ defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)

1.4.9. **Lorentz transformations in $M_2$**

Consider the 2-dimensional Minkowski space $M_2$ with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $2 \times 2$ matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves $g$ invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma, \tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \tau \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \rightarrow D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix $J$ (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine $J$ explicitly.

(6 points)

1.4.10. **Time-like and space-like intervals**

Consider two points $(ct_x, x^1, x^2, x^3)$ and $(ct_y, y^1, y^2, y^3)$ in Minkowski space. The interval between the two points is called *time-like* if

$$c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2,$$

and *space-like* if

$$c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2.$$ 

Show that in interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)
1.4.11. Special Lorentz transformations in $M_4$

Consider the Minkowski space $M_4$.

a) Show that the following transformations are Lorentz transformations:

i) \( D^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix} \equiv R^{\mu}_{\nu} \) (rotations)

where \( R_{ij} \) is any Euclidian orthogonal transformation.

ii) \( D^{\mu}_{\nu} = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^{\mu}_{\nu} \) (Lorentz boost along the $x$-direction)

with \( \alpha \in \mathbb{R} \).

iii) \( D^{\mu}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^{\mu}_{\nu} \) (parity)

iv) \( D^{\mu}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^{\mu}_{\nu} \) (time reversal)

b) Let \( L \) be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of \( L \), and so are the Lorentz boosts defined in part a) ii).

c) Let \( I^{\mu}_{\nu} = \delta^{\mu}_{\nu} \) be the identity transformation. Show that the sets \( \{I, P\} \), \( \{I, T\} \), and \( \{I, P, T, PT\} \) are subgroups of \( L \).

(4 points)