1.4.12. **General Lorentz transformations in \( M_4 \)**

Let \( D \) be a general Lorentz transformation in \( M_4 \).

a) Show that \( |D_0^0| \geq 1 \), and that \((D_1^0)^2 + (D_2^0)^2 + (D_3^0)^2 = (D_0^1)^2 + (D_0^2)^2 + (D_0^3)^2\).

b) Let \( L_{++} = \{ D \in L : \det D > 0, D_0^0 > 0 \} \). (This is called the set of proper orthochronous Lorentz transformations, and one can show that it is a subgroup of \( L \).) Show that any Lorentz transformation can be written as an element of \( L_{++} \) followed by either \( P \), or \( T \), or \( PT \). It thus suffices to study \( L_{++} \).

c) Show that any element of \( L_{++} \) can be written as a spatial rotation \( R(\Phi, \Theta, \Psi) \) followed by a Lorentz boost \( B(\alpha) \) followed by a rotation about the 3-axes followed by a rotation about the 2-axis. In a symbolic notation:

\[
D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_2(\phi)R_3(\theta) & 0 \\ 0 & 0 & R(\Phi, \Theta, \Psi) \end{pmatrix} B(\alpha) \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_2(\phi)R_3(\theta) & 0 \\ 0 & 0 & R(\Phi, \Theta, \Psi) \end{pmatrix}
\]

\( L_{++} \) is thus characterized by six parameters: 3 Euler angles \( \Phi, \Theta, \Psi \), the boost parameter \( \alpha \), and two additional rotation angles \( \phi \) and \( \theta \).

(7 points)

1.5.1. **Transformations of tensor fields**

a) Consider a covariant rank-\( n \) tensor field \( t_{i_1...i_n}(x) \) and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how \( \tilde{t}_{i_1...i_n}(\tilde{x}) \) is related to \( t_{i_1...i_n}(x) \).

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector \( x_i \) (the case \( n = 1 \)), and (ii) the covariant components of the metric tensor \( g_{ij} \).

(4 points)

1.5.2. **Curl and divergence**

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

1.5.3. **Tensor products, and tensor traces**

Prove Propositions 1 and 2 from ch. 1 §5.3.

(4 points)