STATISTICAL MECHANICS

614 Final Exam

6/09/2022

Closed-book exam; please no notes, books, internet connection, etc. Calculators are not necessary.

Ultrarelativistic quantum gases

Consider an ideal Fermi gas which has a momentum distribution

$$\rho(p) = \frac{gV}{2\pi^2\hbar^3}\,\frac{p^2}{e^{\beta(\epsilon_p-\mu)}+1}$$

where $\beta = 1/T$ (use units such that $k_B = 1$), and $\epsilon_p = c\,p$ with c the speed of light (ultrarelativistic case).

a) What is the energy distribution $\rho(\epsilon)$?

b) Show that the grand canonical thermodynamic potential

$$J = -T\sum_p \log\left[1 + e^{\beta(\mu-\epsilon_p)}\right]$$

has the form

$$J = VT^4 f(\mu/T)$$

and express the function f in terms of an integral.

c) Show that the entropy per particle is a function of μ/T only,

$$S/N = \varphi(\mu/T)$$

and express φ in terms of f and its derivative.

hint: Show that $S = VT^3 g(\mu/T)$ and $N = VT^3 h(\mu/T)$, and express g and h in terms of f and f'.

d) Show that for an adiabatic process one has (for fixed particle number N)

$$VT^3 = \text{const.}$$

and

$$pV^{4/3} = \text{const.}$$

e) Show that the corresponding adiabatic equations for a photon gas are the same.

f) The cosmic microwave background radiation can be considered as blackbody radiation in an adiabatically expanding universe of radius R, volume $V = R^3$, and mass M. Use the results of part e) above to show that the ratio of the radiation energy density $u = U/V$ to the mass density $\rho = M/V$ decreases as

$$u/\rho \propto 1/R$$

This is important in cosmology.

(21 points)