Ultrarelativistic quantum gases

Consider an ideal Fermi gas which has a momentum distribution

$$\rho(p) = \frac{gV}{2\pi^2\hbar^3} \frac{p^2}{e^{\beta(\epsilon_p - \mu)} + 1}$$

where $\beta = 1/T$ (use units such that $k_B = 1$), and $\epsilon_p = cp$ with $c$ the speed of light (ultrarelativistic case).

a) What is the energy distribution $\rho(\epsilon)$?

b) Show that the grand canonical thermodynamic potential

$$J = -T \sum_p \log [1 + e^{\beta(\mu - \epsilon_p)}]$$

has the form

$$J = V T^4 f(\mu/T)$$

and express the function $f$ in terms of an integral.

c) Show that the entropy per particle is a function of $\mu/T$ only,

$$S/N = \varphi(\mu/T)$$

and express $\varphi$ in terms of $f$ and its derivative.

**hint:** Show that $S = V T^3 g(\mu/T)$ and $N = V T^3 h(\mu/T)$, and express $g$ and $h$ in terms of $f$ and $f'$.

d) Show that for an adiabatic process one has (for fixed particle number $N$)

$$V T^3 = \text{const.}$$

and

$$p V^{4/3} = \text{const.}$$

e) Show that the corresponding adiabatic equations for a photon gas are the same.

f) The cosmic microwave background radiation can be considered as blackbody radiation in an adiabatically expanding universe of radius $R$, volume $V = R^3$, and mass $M$. Use the results of part e) above to show that the ratio of the radiation energy density $u = U/V$ to the mass density $\rho = M/V$ decreases as

$$u/\rho \propto 1/R$$

This is important in cosmology.