PHYS 614  Spring 2022

Note: This version of 614 builds on 613 as taught by Jayson Paulose in Winter 2022. The 613 table of contents was as follows:

Lecture 1: Probability (single variable)
- Definitions
- Moments and cumulants and their generating functions
- Binomial, Gaussian, Poisson distributions

Lecture 2: Probability (multiple variable)
- Marginal and conditional probabilities
- Bayes' theorem
- Central limit theorem

Lecture 3: Classical statistical mechanics from phase space dynamics
- Liouville's theorem
- Ergodicity
- Fundamental postulate of classical stat mech
- Phase space volume and phase space probability density in the microcanonical ensemble

Lecture 4: Entropy and equilibrium
- Temperature via maximizing combined entropy
- Saddle-point approximation
- Second law of thermodynamics
- Stability of equilibrium and connection to heat capacity

Lecture 5: Two-state system (noninteracting binary spins with energy difference)
- Entropy via binomial distribution and Stirling's approximation
- Energy, temperature, heat capacity
- Microstate probabilities as a function of temperature and hints of canonical ensemble

Lecture 6: Thermodynamics from entropy
- Chemical potential and pressure from maximizing combined entropy
- Thermodynamic identity and evaluation of partial derivatives
- First law of thermodynamics (energy conservation)
- Connecting entropy changes to heat and work; quasistatic processes
- Generalized forces and relation to constrained quantities
- Cycles, heat engines, efficiency

Lectures 7-8: Canonical ensemble
- Definition in terms of entropy maximization of system + bath
- Partition function, Helmholtz free energy
- Equivalence of microcanonical and canonical ensembles
- Partition function of the two-state system

Lecture 8: Ideal gas
- Evaluation of phase space volume in microcanonical ensemble
- Sackur-Tetrode equation for entropy
- Maxwell-Boltzmann distribution
- Partition function of ideal gas

Lecture 9: Other statistical ensembles
- Legendre transforms to swap dependences of intensive and extensive quantities
- Gibbs ensemble (V \rightarrow p)
- Grand canonical ensemble (N \rightarrow \mu)
- Gibbs-Duhem relation
Chapter 5
Principles of Stochastic Mechanics (will explore in 6.7)

§1 Stochastic description of open systems

1.1 Flow of Hamiltonian systems

With a non-autonomous, closed, undamped system with \( f \) degrees of freedom, let the generalized coordinates and momenta be

\[
q(t) = (q_1(t), \ldots, q_f(t))
\]

\[
p(t) = (p_1(t), \ldots, p_f(t))
\]

**Def. 1**: The phase space \( \Gamma \) of the system is the set of points

\[
\mathcal{T}(t) = (q(t), p(t))
\]

is a \( 2f \)-dimensional space.

**Remark**: (1) Given an initial admissible \( \mathcal{T}(t_0) \), Hamiliton's

epi uniquely determines \( \mathcal{T}(t) \) for \( t > t_0 \) and

\( \forall t < t_0 \).

**Lemma**: Liouville's Theorem

Let \( \mathcal{D} \in \Gamma \) be a bounded region, and consider its time

evolution \( \Delta(t) \) according to Hamilton's
eqs. Let \( V(t) \) be a volume of \( \Delta(t) \).

Then \[
V(t) = V(t=0) = \text{const}
\]

**Remark**: (2) Phase flow is a volume preserving mapping.

**Proof**: PHYS 612
Definition 2.1: Let $G = \{ f_x : x \in \mathbb{R} \}$ be a set of mappings. $G$ is called a one-parameter group of transformations if:

1. $G$ is a group under successive mappings, i.e.,
   
   \[
   f_{x+y} = f_x f_y \quad \text{and} \quad f_{x+y} = f_{y+x}
   \]

   Remark: (1) All reid groups are isomorphic to $\mathbb{R}(+)$.

2. The operator $U$ that describes plane flow via

   \[
   f(t) = U(t-t_0) f(t_0)
   \]

   is called the time evolution operator.

Definition 2.2: The set of time evolutions $\{ U(t), t \in \mathbb{R} \}$ forms a one-parameter group under successive time evolutions.

Proof: Phys. 612

Remark: (4) Time evolution is continuous and invertible.

1.2 Poincare's Theorem

Theorem: Poincare's Recurrence Theorem

Let $G$ be a group of volume-preserving, continuously invertible mappings of $\mathbb{R}^n$ onto itself. Let $D$ be a bounded region in $\mathbb{R}^n$ with $G$-invariant measure. Then for any region $U \subset D$ and for any $g \in G$ there exists an $n \in \mathbb{N}$ such that $g^n(U) \cap D = \emptyset$. Therefore, $g^n(U) \cap D = \emptyset$ for all $n \in \mathbb{N}$.
proof: Assume the claims of \( \{g^n, 0 \leq n \leq N\} \) are pairwise disjoint, i.e., \( g^n \cap g^m = \emptyset \) for \( n \neq m \) and \( N \in \mathbb{N} \).

\[ \forall n \in \mathbb{N}: g^n \cap g^m = \emptyset \text{ for } n \neq m \]

\[ N+1 \leq \text{Vol}(D) / \text{Vol}(U) \tag{8} \]

Then
\[ N = \inf \left( \frac{\text{Vol}(D)}{\text{Vol}(U)} \right) \]

\[ N + 1 > \frac{\text{Vol}(D)}{\text{Vol}(U)} \]

But this contradicts (8).

\[ \exists N \in \mathbb{N}: \text{the claims of } \{g^n, 0 \leq n \leq N\} \text{ are not pairwise disjoint} \]

\[ \exists x, y \in \mathbb{R}, 0 \leq x < y \leq N: g^x \cap g^y \neq \emptyset \]

\[ \exists x, y \in \mathbb{R}: g^x \cap g^y \neq \emptyset \]

\[ g^{x+y} = g^x \cap g^y \]

\[ \text{Union } \forall \Rightarrow \]

\[ x, y \in \mathbb{R} \Rightarrow g^x \cap g^y \neq \emptyset \]
Define $V = \mathcal{G}^{-n}(V) + \emptyset$.

$$
\text{mark: (2) important points:} \\
\text{GD CA } \Rightarrow \text{ image can't} \\
\text{fit out of D}
$$

**Problem 1** (plan 200)

**Problem 2** (idea: experts)

**Problem 3** (true: experts)

**Problem 4** (proof of 2) (4) This is a purely geometric theorem. For physical applications, let

$$
D = \text{hypersphere} \Rightarrow \text{allowed by energy constraints} \\
U = \text{E-neighborhood of some point } t \in D.
$$

**Example:** Let a bounded system of him to be a steel join by $\tau \in \mathbb{R}$. Run with a finite count of time it come vibrating done ogni to $t_0$.

- $t_0$
- $t > t_0$
- $\tau > t_0$
1.5 Numerical verification of a stochastic discipline of non-potential fields

Under the evolution of field maps in plan space according to bowers's law, Poisson's law.

Examples:

1. For plane $d=1$

\[ p(t) = p_0 \]
\[ q(t) = q_0 + (p_0 - 1) t \]

2. Plane $d=1$ polyhedron

\[ p(t) = p_0 \]
\[ q(t) = q_0 + (p_0 - 1) t \]

In this case, the plane becomes disconnected.

As $t \to 0$, disconnected parts become thinner, but the overall structure remains.

Demonstrated: Poisson's portrait

(2) Plane: Method by Lord Kelvin for $d=2$

\[ b/5 = v \times \]
\[ d_q = d_q' \]

> $q$ - Uniqueness to plan space

$\phi$ - Potential function
\[ u \ll 1 \Rightarrow \Delta x = \Delta b / \tau = \Delta q / \tau \]

Then for \( \Delta x \gg \tau \) two spatial volume \( \frac{a}{h} \) will combine:

\[
V' = \Delta q \cdot b \cdot \Delta x = \Delta q \cdot b \cdot \Delta q / \tau \cdot \Delta x / \tau = V \left( \frac{\tau^2}{\Delta q} \right)
\]

\( V \) : d-dimensional

\( V' \sim V \left( \frac{\tau}{\Delta q} \right)^d \)

Woof is a nontrivial contribution to phase space volume loss demand by a factor of \( \left( \frac{\tau}{\Delta q} \right)^{d-1} \) after \( N \) collisions.

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**Problem:**

"Butterfly Effect" (Flood)

After \( N \) collisions, a right volume \( \Gamma \) loss be reformed into a foliation structure will have dimension:

\[
(d \rho / \rho) \sim \left( \frac{\tau}{\Delta q} \right)^{d-1} N
\]

---

**Example:** derived yes of local volume will reduce \( 5 \),

\[
\frac{\Delta x}{\tau} \Rightarrow b = 1 / \Delta q \tau
\]

\[
\left( \frac{\tau}{\Delta q} \right) \sim 10^{-2} \sim \text{a typical yes}
\]

\[
1+2, N-2 : (d \rho / \rho) \sim \left( 10^{-2} \right)^{1+2} \sim 10^{-18} (?)
\]

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After a few collisions, nothing seems to maintain the foliation structure!

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**Remark:** (1) red experiment: violent force is roughly \( E / m \sim \frac{G m^2}{r^2} \cdot 10^{-8} \)

mass uncertainty due to uncontrolled forces:

\[
\Delta \rho \sim E / \tau \Rightarrow \Delta \rho / \rho \sim \frac{E}{\tau \rho} \sim \frac{E}{\tau}
\]

derived yes: \( v = 10^5 \text{ m/s}, \tau = 10^{-9} \) typically:

\[
\Delta \rho / \rho \sim 10^{-21} \text{ m/s} \sim (\text{m}^2)
\]
(i) \( N = 1.9 \) at distance \( r = 10^6 \text{m} \) (butterfly outside lab)
\[ \Rightarrow \Delta \rho / \rho \approx 10^{-79} \]
\[ \Rightarrow \text{fohich structure destroyed after } N \approx \frac{\Delta \rho}{\rho} \approx 5 \text{ collisions} \]

(ii) \( N = 1.9 \) at distance \( r = 1 \text{pc} = 3 \times 10^{16} \) m (butterfly on hi"n)
\[ \Rightarrow \Delta \rho / \rho \approx 10^{-54} \]
\[ \Rightarrow \text{fohich structure destroyed after } N \approx \frac{\Delta \rho}{\rho} \approx 9 \text{ collisions} \]

(2) Computer experiment: many error introduses unmeetable perturbations.

Conclusion: (1) Microscopically deterministic discipline of \( N \)-particle

\[ \text{eules (like him nonability, Paican's lemma, etc.)} \]
\[ \text{is responsible, now or later no control over butterfly} \]
\[ \text{on wire} \]

(2) Exponet to him errors of macroscopic particle

\[ \text{on xpechiplet of him disturbance} \]

(3) Develop statistical (rather than deterministic)

\[ \text{discipline of many-particle eules} \]