remark: (3) $\chi_{dia} = -\frac{1}{3}\chi_{para} \rightarrow \chi = \chi_{dia} + \chi_{para} = \frac{2}{3}\chi_{para} > 0$

(4) external B-field has two effects:

(a) it tends to align the spins $\rightsquigarrow$ paramagnetism

(b) it induces a diamagnetic (Lenz's rule!) current due to the orbital motion of the electrons $\rightsquigarrow$ diamagnetism

(5) For an ideal electron gas at $T=0$, $\partial n/\partial\mu$, the specific heat, χ_{para}, χ_{dia}, and the density of states are closely related: With $\omega(\varepsilon)$ the DOS per unit volume we have

$$n = \frac{N}{V} = \int_0^\infty d\varepsilon\, \omega(\varepsilon) f_\mu(\varepsilon) \overset{T=0}{=} \int_0^{\varepsilon_F} d\varepsilon\, \omega(\varepsilon)$$

$$\rightsquigarrow \quad \underline{\underline{\frac{\partial n}{\partial \mu}}} = \omega(\varepsilon_F) =: \underline{\underline{N_F}} \quad \text{DOS at the Fermi level}$$

$$= \frac{3}{2} n/\varepsilon_F$$

$$\S 2.3 \rightsquigarrow \underline{C_V/V} = n k_B \frac{\pi^2}{2} k_B T/\varepsilon_F = \underline{\underline{\frac{\pi^2}{3} k_B N_F \cdot k_B T}} =: \underline{\gamma T}$$

$$\S 2.3,4 \rightsquigarrow \underline{\underline{\chi_{para} = -3\chi_{dia} = \mu_B^2 N_F}}$$

$\underline{\underline{\gamma = \frac{\pi^2}{3} k_B^2 N_F}}$ specific heat coefficient

§3 The ideal Bose gas

3.1 Bose-Einstein condensation

Consider an ideal Bose gas. cf. §5.5 $\rightsquigarrow$

$$N = \sum_k n_\mu(\varepsilon_k) = \sum_k \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} = \frac{1}{e^{\beta(\varepsilon_0-\mu)}-1} + \sum_{k\neq 0} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1}$$

$\rightarrow$

$N = N_0 + N'$	total particle #
$N_0 = \frac{1}{e^{-\beta\mu}-1}$	# of particles in ground state
$N' = \sum_{k\neq 0} \frac{1}{e^{\beta(\varepsilon_k-\mu)}-1} \overset{\S 2.1}{=} \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty d\varepsilon \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}-1}$	# of particles in excited states

remark: (1) We have put $\varepsilon_0 = 0$ w.l.g. (otherwise, just redefine μ)

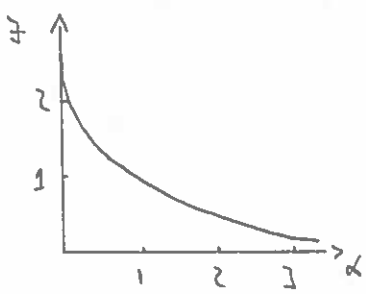
(2) §2.1 remark (4') $\rightarrow$ $\omega(\varepsilon) \sim \varepsilon^{1/2}$ $\rightarrow$ in going from a mono to a ideal, the ground state gets zero weight.

(3) $N_0(\mu \to 0) \to \infty$, and problem #72 $\rightarrow$ $\mu(T \to T_0) \to 0$ with $T_0 > 0$ $\rightarrow$ neglecting N_0 may be dangerous!

$$\underline{\underline{N'}} = \frac{\cancel{gV}(m k_B T)^{3/2}}{\cancel{\sqrt{2}\pi^2 \hbar^3}} \frac{\sqrt{\pi}}{\cancel{2}} \frac{2}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^{1/2}}{e^x e^{-\lambda\mu} - 1} = \underline{gV \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \mathcal{F}(-\lambda\mu)}$$

where $\underline{\mathcal{F}(\alpha) := \frac{2}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^{1/2}}{e^x e^\alpha - 1}}$ $\quad (\alpha > 0)$

discussion of $\mathcal{F}(\alpha)$: $\quad \mathcal{F}(0) = \frac{2}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^{1/2}}{e^x - 1} = \frac{2}{\sqrt{\pi}} \Gamma(3/2) \underbrace{\zeta(3/2)}_{\text{Riemann } \zeta\text{-fct.}} = \zeta(3/2) = 2.612.$



$e^\alpha > 1 \;\; \forall \alpha > 0 \rightarrow \mathcal{F}(\alpha)$ decreases monotonically with increasing α

$$\mathcal{F}(\alpha \to \infty) \to \frac{2}{\sqrt{\pi}} e^{-\alpha} \int_0^\infty dx \, x^{1/2} e^{-x} = e^{-\alpha}$$

$$\rightarrow \quad N' \leq N'_{max} = gV \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \mathcal{F}(0) = gV \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \zeta(3/2)$$

define T_0 by

$$gV \left(\frac{m k_B T_0}{2\pi\hbar^2}\right)^{3/2} \zeta(3/2) = N$$

$$\rightarrow \quad \underline{k_B T_0} = \frac{2\pi\hbar^2}{m} \left(\frac{N}{V}\right)^{2/3} \frac{1}{(g\,\zeta(3/2))^{2/3}} = \underline{\frac{a}{g^{2/3}} \frac{\hbar^2}{m} n^{2/3}}, \quad \underline{a = 2\pi / (\zeta(3/2))^{2/3}} = \underline{3.3125\ldots}$$

$\rightarrow$ For $T < T_0$, the excited states cannot accomodate all of the particles!

remark: (4) Cf. problem #72

proposition: In an ideal Bose gas at $T < T_0$, with

$$\boxed{k_B T_0 = \frac{a}{g^{2/3}} \frac{\hbar^2}{m} n^{2/3}} \qquad \underline{a = 2\pi / (\zeta(3/2))^{2/3} = 3.31\ldots} \qquad \underline{n = N/V}$$

a finite fraction N_0/N of all particles are in the ground state, even for $N \to \infty$.

remark: (5) Problem #31 $\rightsquigarrow$ T_0 is the temperature where $\mu(T \to T_0+) \to 0-$

(6) The accumulation of a finite density of particles in the ground state is called Bose-Einstein condensation

3.2 Chemical potential and particle number for $T < T_0$

$$\#31 \rightsquigarrow \quad N_0 = N - N' = \frac{1}{e^{-\beta\mu} - 1} \quad \rightsquigarrow \quad e^{-\beta\mu} = 1 + \frac{1}{N - N'}$$

$$\rightsquigarrow \quad \underline{-\mu} = k_B T \log\left(1 + \frac{1}{N - N'}\right) \overset{T<T_0}{=} \underline{\underline{k_B T \log\left(1 + \frac{1}{N - N'_{max}}\right)}}$$

remark: (1) For finite N, $\mu(T<T_0)$ assumes a small negative value so that the correct number $N_0 = N - N'_{max}$ of particles is accomodated in the ground state.

(2) In the thermodynamic limit, $N \to \infty$ while $N'_{max} = \text{const}$

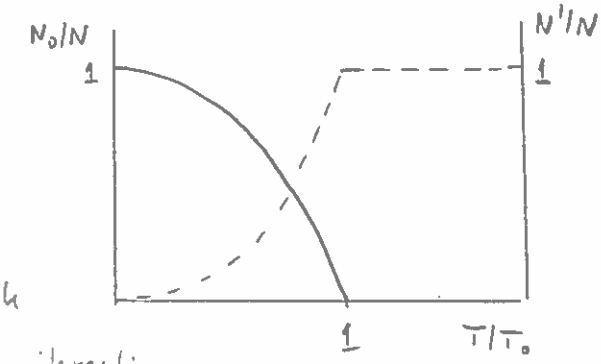
$$\rightsquigarrow \quad N - N'_{max} = \infty \quad \rightsquigarrow \quad \underline{\underline{\mu(T<T_0) = 0}}$$

Consider the T-dependence of N' and N_0:

$$\underline{N'} \overset{\#31}{=} g V \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} \underset{\downarrow\, T<T_0 \to \mu=0}{\zeta(0)} = g V \left(\frac{m k_B T_0}{2\pi\hbar^2}\right)^{3/2} \zeta(0) (T/T_0)^{3/2}$$

$$\overset{\#31}{=} \frac{N}{\zeta(3/2)} \zeta(0) (T/T_0)^{3/2} = \underline{N (T/T_0)^{3/2}}$$

$$\rightarrow \quad \boxed{N'/N = (T/T_0)^{3/2}} \quad \boxed{N_0/N = 1-(T/T_0)^{3/2}}$$



remark: (3) n, m for He 4

$\rightarrow T_0 = 3.13\,K$

(4) He 4 has indeed a phase transition at $T_c = 2.2\,K$

problem: He 4 is strongly interacting

(5) Rarely, BEC has been observed in very weakly interacting systems.

3.3 The specific heat of an ideal Bose gas

$T < T_0$: §3.2 $\rightarrow \mu = 0$

$$\rightarrow \quad U = \sum_r \varepsilon_r n_\mu(\varepsilon_r) = \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty d\varepsilon\, \frac{\varepsilon^{3/2}}{e^{\beta\varepsilon}-1}$$

$$= \frac{gV(m k_B T)^{3/2}}{\sqrt{2}\pi^2\hbar^3}\, k_B T\, \frac{\sqrt{\pi}}{2}\,\frac{2}{\sqrt{\pi}} \underbrace{\int_0^\infty dx\, \frac{x^{3/2}}{e^x-1}}_{\Gamma(5/2)\zeta(5/2)}$$

$$\overset{\S 3.2}{=} \frac{N}{\zeta(3/2)}(T/T_0)^{3/2}\, k_B T\, \frac{2}{\sqrt{\pi}}\,\frac{3}{2}\,\frac{\sqrt{\pi}}{2}\,\zeta(5/2)$$

$$= \underline{N k_B T\,(T/T_0)^{3/2}\,\tfrac{3}{2}\,\zeta(5/2)/\zeta(3/2)} = \underline{0.77\ldots \times N k_B T\,(T/T_0)^{3/2}} =: U_0(T)$$

$$\rightarrow \quad C_V = \frac{\partial U}{\partial T} = \frac{5}{2}\,U/T = \underline{N k_B (T/T_0)^{3/2}\,\tfrac{15}{4}\,\zeta(5/2)/\zeta(3/2)} \qquad (T<T_0)$$

$T > T_0$: Now we need $\mu(T)$!

Since: $$\boxed{\mu(T\to T_0+) = -\frac{(\zeta(3/2))^2}{4\pi}\, k_B T_0 \left[(T/T_0)^{3/2}-1\right]^2}$$

proof: $$\underline{N} = \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty d\varepsilon\,\frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)}-1} = \underline{\frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty d\varepsilon\,\varepsilon^{1/2}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}-1}-\frac{1}{e^{\beta\varepsilon}-1}\right)} + N''(T)$$

where $$\underline{N''(T)} := \frac{gVm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\int_0^\infty d\varepsilon\,\frac{\varepsilon^{1/2}}{e^{\beta\varepsilon}-1} \overset{\S 2.2}{=} \underline{N(T/T_0)^{3/2}}$$

remark: (1) For $T<T_0$, N'' would be N', the number of particles in the excited states.

$T \to T_0+ \rightsquigarrow \mu \to 0 \rightsquigarrow$ the leading contribution to the ε-integral comes from $\beta\varepsilon \ll 1$, $\beta(\varepsilon-\mu) \ll 1$.

$$\rightsquigarrow \underline{\int_0^\infty d\varepsilon\,\varepsilon^{1/2}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}-1}-\frac{1}{e^{\beta\varepsilon}-1}\right)} \simeq \int_0^\infty d\varepsilon\,\varepsilon^{1/2}\left(\frac{1}{\beta(\varepsilon-\mu)}-\frac{1}{\beta\varepsilon}\right)$$

$$= k_B T\int_0^\infty d\varepsilon\,\frac{\mu\,\varepsilon^{1/2}}{\varepsilon(\varepsilon-\mu)} = k_B T(-\mu)^{1/2}(-)\underbrace{\int_0^\infty dx\,\frac{1}{\sqrt{x}\,(x+1)}}_{=\pi}$$

$$= \underline{-\pi k_B T\sqrt{-\mu}}$$

$$\rightsquigarrow N = \frac{-gVm^{3/2}\pi}{\sqrt{2}\pi^2\hbar^3}k_B T\sqrt{-\mu} + N(T/T_0)^{3/2}$$

$$\rightsquigarrow \underline{-\mu} = \left(\frac{N}{V}\right)^2\left[1-(T/T_0)^{3/2}\right]^2\frac{2\pi^4\hbar^6}{g^2m^3\pi^2}\frac{1}{(k_BT)^2}$$

$$\overset{T\to T_0}{=} \frac{2\pi^2\hbar^6}{g^2m^3(k_BT_0)^2}\left[1-(T/T_0)^{3/2}\right]^2 \overset{\S 2.1\ \text{prop.}}{=} \frac{2\pi^2}{c^2}k_BT_0\left[1-(T/T_0)^{3/2}\right]^2$$

$$= \underline{\underline{\frac{1}{4\pi}(\zeta(3/2))^2 k_BT_0\left[1-(T/T_0)^{3/2}\right]^2}}$$

$\S 2.1 \rightsquigarrow$ $$\mathcal{J} = -pV = -\frac{2}{3}U$$

$$\rightsquigarrow \frac{\partial U}{\partial\mu} = -\frac{3}{2}\frac{\partial\mathcal{J}}{\partial\mu} = +\frac{3}{2}N$$

$$\rightsquigarrow \underline{U} = U(\mu=0) + \frac{3}{2}N\mu = \underline{U_0(T) + \frac{3}{2}N\mu(T)}$$

$$\rightsquigarrow \underline{C_V} = \frac{\partial U}{\partial T} = \underline{\frac{\partial U_0}{\partial T} + \frac{3}{2}N\frac{\partial\mu}{\partial T}}$$

remark: (2) We know $\partial U_0/\partial T$ from the calculation for $T<T_0$:

$$\partial U_0/\partial T = N k_B (T/T_0)^{3/2} \frac{15}{4} \zeta(5/2)/\zeta(3/2) .$$

For $T<T_0$ this is C_V, for $T>T_0$ it is just some contribution to C_V

$$\rightsquigarrow \quad C_V(T\to T_0+) = \frac{\partial U_0}{\partial T} - \frac{\cancel{2}}{\cancel{\lambda}} N \frac{\lambda \cancel{\pi^2} \cancel{k^6} \cancel{\mu^2}}{\cancel{g^2} \cancel{\mu^2} (k_B T_0)^2} \lambda \left[1-(T/T_0)^{3/2}\right] \frac{\cancel{\lambda}}{\cancel{2}} \frac{T^{1/2}}{T_0^{3/2}}$$

$$= \cancel{\frac{\partial U_0}{\partial T}} + \frac{9\pi^2 \cancel{k^6} \cancel{\mu^2}}{\cancel{g^2} \cancel{\mu^2}} \frac{N k_B}{(k_B T_0)^2} \left[1-(T/T_0)^{3/2}\right]$$

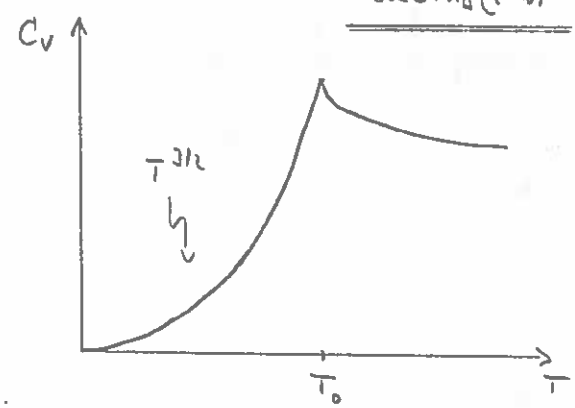
$$= N k_B \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} \left[\cancel{1} + \left((T/T_0)^{3/2}-1\right)\right] - \frac{9}{2\pi} \left(\zeta(3/2)\right)^2 N k_B \left[(T/T_0)^{3/2}-1\right]$$

cf. LL §62 Problem ✓

$$= C_V(T_0) - N k_B \left[\frac{9}{2\pi}\left(\zeta(3/2)\right)^2 - \frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}\right]\left[(T/T_0)^{3/2}-1\right] = C_V(T_0) - 0.52\, N k_B \left[(T/T_0)^{3/2}-1\right]$$

remark: (3) $C_V(T)$ is continuous, but not differentiable at $T=T_0$: cusp-singularity.

(4) In 4He, $C_V(T)$ diverges at $T=T_c$ (albeit very weakly), the 'lambda-point'). Interactions lead to a stronger singularity!



§4 Blackbody radiation

4.1 Planck's law

Consider photons in equilibrium with matter. Photons have spin 1 ⇝ B-E statistics applies.

remark: (1) Photon-photon interaction is extremely weak (mediated only by virtual $e^- - e^+$ pairs) ⇝ some matter is necessary to achieve equilibrium.

(2) Photon gas can be considered ideal if the matter is, e.g., a dilute gas, or the walls of a cavity whose only role is to absorb