Problem Assignment # 2

04/07/2022
due 04/14/2022

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5. Integrated density of states

The number of accessible states, $\Omega(E)$, is defined as the number of states with energies between $E - \Delta E$ and E. Alternatively, we can define $\tilde{\Omega}_x(E)$ as the number of states with energies between xE and E, where $0 \leq x \leq 1$. For a classical system,

$$\tilde{\Omega}_x(E) = const. \times \int_{xE \leq H \leq E} d\Gamma \quad ,$$

where $d\Gamma = d^3x_1 \ldots d^3x_N \, d^3p_1 \ldots d^3p_N$ is the phase space volume element, and H is the energy of a microstate. The normalization constant will be of no relevance for what follows. $\tilde{\Omega}_{x=0}(E)$ is called the *integrated density of states.*

a) For a classical ideal gas (N noninteracting point particles of mass m in a volume V), show that

$$\tilde{\Omega}_{x=0}(E) = const. \times V^N \, (2mE)^{3N/2} \, C_{3N} \quad ,$$

with C_d the volume of the d-dimensional unit sphere. Calculate C_d.

b) Show that

$$\tilde{\Omega}_x(E) = f(x) \, \tilde{\Omega}_0(E) \quad ,$$

and determine $f(x)$. How close to 1 do you have to choose x in order for $f(x)$ to be substantially different from unity? Discuss the meaning of this result for the volumes of high-dimensional spheres, and for the physical significance of the arbitrary energy interval ΔE in the number of accessible states.

(7 points)

6. Particle in a box

Consider a quantum mechanical system consisting of one spinless particle in a 3-dimensional rectangular box with linear dimensions L_1, L_2, and L_3.

a) Suppose the system is in a particular microstate. From the change of the corresponding energy level under a quasi-static change of the length L_i by dL_i, find the force exerted by the particle on the wall perpendicular to the i-axis.

b) For a cubic box, find the average pressure of the particle on a wall in terms of the average energy of the particle and the volume of the box.

hint: You do not need to find the probability distribution explicitly.

(4 points)

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7. Energy fluctuations

Consider a canonical ensemble with partition function z.

a) Show that the mean energy and the root-mean-square deviation of the energy are given by

$$U = \langle E \rangle = -\partial \log Z / \partial \beta$$

$$(\Delta E) = \left\langle (E - \langle E \rangle)^2 \right\rangle^{1/2} = \left(\partial^2 \log Z / \partial \beta^2 \right)^{1/2}$$

b) Show that the energy distribution function is Gaussian,

$$\rho(E) = \frac{1}{\sqrt{2\pi}\sigma} \, e^{-(\Delta E)^2/\sigma^2}$$

where the standard deviation is small of order

$$\sigma^2 / U^2 = O(1/N)$$

hint: Divide the system into a large number of identical subsystems that are still macroscopic and each have canonical distribution, and use the central limit theorem.

(7 points)