5. Integrated density of states

The number of accessible states, $\Omega(E)$, is defined as the number of states with energies between $E - \Delta E$ and $E$. Alternatively, we can define $\tilde{\Omega}_x(E)$ as the number of states with energies between $xE$ and $E$, where $0 \leq x \leq 1$. For a classical system,

$$\tilde{\Omega}_x(E) = \text{const.} \times \int_{x E \leq H \leq E} d\Gamma,$$

where $d\Gamma = d^3x_1 \cdots d^3x_N \, d^3p_1 \cdots d^3p_N$ is the phase space volume element, and $H$ is the energy of a microstate. The normalization constant will be of no relevance for what follows. $\tilde{\Omega}_x=0(E)$ is called the integrated density of states.

a) For a classical ideal gas ($N$ noninteracting point particles of mass $m$ in a volume $V$), show that

$$\tilde{\Omega}_x=0(E) = \text{const.} \times V^N \left( \frac{2mE}{C_d} \right)^{3N/2},$$

with $C_d$ the volume of the $d$-dimensional unit sphere. Calculate $C_d$.

b) Show that

$$\tilde{\Omega}_x(E) = f(x) \tilde{\Omega}_0(E),$$

and determine $f(x)$. How close to 1 do you have to choose $x$ in order for $f(x)$ to be substantially different from unity? Discuss the meaning of this result for the volumes of high-dimensional spheres, and for the physical significance of the arbitrary energy interval $\Delta E$ in the number of accessible states.

(7 points)

6. Particle in a box

Consider a quantum mechanical system consisting of one spinless particle in a 3-dimensional rectangular box with linear dimensions $L_1$, $L_2$, and $L_3$.

a) Suppose the system is in a particular microstate. From the change of the corresponding energy level under a quasi-static change of the length $L_i$ by $dL_i$, find the force exerted by the particle on the wall perpendicular to the $i$-axis.

b) For a cubic box, find the average pressure of the particle on a wall in terms of the average energy of the particle and the volume of the box.

$\text{hint:}$ You do not need to find the probability distribution explicitly.

(4 points)
7. Energy fluctuations

Consider a canonical ensemble with partition function $z$.

a) Show that the mean energy and the root-mean-square deviation of the energy are given by

$$U = \langle E \rangle = -\partial \log Z / \partial \beta$$

$$\langle \Delta E \rangle = \langle (E - \langle E \rangle)^2 \rangle^{1/2} = \left( \partial^2 \log Z / \partial \beta^2 \right)^{1/2}$$

b) Show that the energy distribution function is Gaussian,

$$\rho(E) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\langle \Delta E \rangle^2 / \sigma^2}$$

where the standard deviation is small of order

$$\frac{\sigma^2}{U^2} = O(1/N)$$

Hint: Divide the system into a large number of identical subsystems that are still macroscopic and each have canonical distribution, and use the central limit theorem.

(7 points)