1. Phase flow

Consider a point mass in \( d = 1 \) subject to a constant force \( f \), so that the equation of motion is

\[
\ddot{q}(t) = \frac{f}{m}.
\]

a) Determine the phase flow. Does Liouville's theorem hold?

b) Does Poincare's theorem hold? Justify your answer.

(4 points)

\[\begin{align*}
\dot{q}(t) &= \frac{f}{m} \\
\Rightarrow \quad \dot{q} - \dot{q}_0 + \left(\frac{f}{m}\right) t = \rho = m q = \rho_0 + ft
\end{align*}\]

\[
\begin{pmatrix}
q(t) \\
p(t)
\end{pmatrix} = \begin{pmatrix}
\frac{f}{m} t^2 & t \\
ft & 1
\end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}
\]

\[\begin{align*}
t(t) &= \begin{pmatrix} \frac{f}{m} t^2 \\ ft \end{pmatrix} + D(t) t_0, \quad \text{with} \quad D(t) = \begin{pmatrix} 1 \\ t/m \end{pmatrix}^t \\
\text{disturbion:} \quad dq \text{ increases linear with } t \\
\text{hypothesis:} \quad dp \text{ is constant}
\end{align*}\]

\[
\det D(t) = 1
\]

\[
\Rightarrow \text{ Liouville's theorem holds}
\]

b) Poincare's theorem obviously does not hold.

Remark: This is not a mapping of a bounded region onto itself!
2. Circle translations

Let $S_1$ be the 1-sphere (a.k.a. circle), and let $g_\alpha$ be a rotation of $S_1$ about its center through an angle $\alpha$. Show that

a) There exists an $n \in \mathbb{Z}$ so that any point $x \in S_1$ returns to its original position after $n$ rotations if and only if $\alpha/2\pi \in \mathbb{Q}$.

b) If $\alpha/2\pi \notin \mathbb{Q}$, then the set of points $\{g^k x; k \in \mathbb{Z}\}$ is dense in $S_1$.

**Notation:** $\mathbb{Z} =$ integers, $\mathbb{Q} =$ rationals.

**Solution:**

$$S_1 = \{ x; x \in \mathbb{R} \pmod{2\pi} \}$$

$$g_\alpha : S_1 \to S_1 \quad \forall x \in S_1, \quad g_\alpha x = x + \alpha \pmod{2\pi}$$

For $x \in S_1$, define

$$N_x = \{ g_\alpha^n x; n \in \mathbb{Z} \}$$

**c)** Let $\alpha/2\pi \in \mathbb{Q} \Rightarrow \exists m, n \in \mathbb{Z}: \alpha = m/n \Rightarrow g_\alpha^n x = x + \alpha \left( \frac{m}{n} \right) \pmod{2\pi} \Rightarrow g_\alpha^m x = x \Rightarrow x \in S_1 \Rightarrow x \in \mathbb{Q}$

**b)** Let $\alpha/2\pi \notin \mathbb{Q}$. a) $N_x$ is uncountable, and let $p_n$ be the $n$th distinct point in $N_x$.

Let $x \in S_1$, let $U_{\varepsilon/2}(x)$ be $\varepsilon/2$-neighborhood of $x$. $p-$known $\Rightarrow \exists n > 1$ and a VCU: $g^m V \subseteq U$

Let $x_0 \in V$. $\Rightarrow 0 < r := |g^n(x - x_0)| < \frac{\varepsilon}{2}$

$$|g^n(x - x_0)| \leq |g^n(x - x_0)| + |g^n(x_0 - x_0) + (x_0 - x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Let $y \in S_1$ arbitrary, fixed $\varepsilon$, $|g^m(x - y)| = 2\pi$
\[ \exists k \in \mathbb{N} : \frac{k-1}{d} \leq \frac{1}{d} < k \]

\[ |g^{kn}x - y| \leq d < \varepsilon \]

For all \( \varepsilon > 0 \) there exists a \( N \) such that \( d < \varepsilon \).
3. Torus translations

Let $T_2 = S_1 \times S_1$ be the 2-torus (i.e., the ordinary doughnut, or the set of points $x = (\phi_1, \phi_2)$, $\phi_i \in \mathbb{R}$ $(i = 1, 2)$, and $(\phi_1, \phi_2) = (\psi_1, \psi_2)$ if and only if $(\phi_i - \psi_i)/2\pi \in \mathbb{Z}$ $(i = 1, 2)$.) Let torus translations be defined by

$$f_t(\phi_1, \phi_2) = (\phi_1 + c_1 t, \phi_2 + c_2 t), \quad t \in \mathbb{R}, \quad 0 \neq c_i \in \mathbb{R}, \quad (i = 1, 2).$$

The winding line $\ell$ through a point $x_0 \in T_2$ is defined as the set

$$\ell = \{x \in T_2; x = f_t(x_0), \ t \in \mathbb{R}\}.$$

Show that

a) $\ell$ is closed if and only if $c_1/c_2 \in \mathbb{Q}$.

b) If $\ell$ is not closed, then $\ell$ is dense in $T_2$.

(4 points)

notation: $\mathbb{R} =$ reals, $\mathbb{Q} =$ rationals, $\mathbb{Z} =$ integers.

[Diagram showing the winding line $\ell$ through a point $x_0$ in the 2-torus $T_2$.]

[Diagrams showing the transformation of points under the translation $f_t$.]

- For $c_1/c_2 \in \mathbb{Q}$:
  - Define angle $x = 2\pi c_1/c_2$.
  - Then $f_x(\psi_1, \psi_2) = (\psi_1 + x, \psi_2)$.
  - In particular, $f_x$ maps $\mathbb{R}_+ = \{(x, 0)\}$ onto $\{(x + \varphi, 0)\}$.

- Now define angle $\beta = 2\pi c_1/c_2$, and $f_\beta : T_2 \to T_2$.

  a) Let $\ell$ be closed if $\Rightarrow \exists \ n \in \mathbb{Z} : g^n x = x \Rightarrow \exists \ m \in \mathbb{Z} : x + m/\beta = x$.

  $\Rightarrow \exists \ n \in \mathbb{Z} : g^n c_1 c_2 = c_1 c_2 \Rightarrow c_1/c_2 \in \mathbb{Q}$.

  b) Let $\ell$ be not closed $\Rightarrow c_1/c_2 \notin \mathbb{Q}$ $\Rightarrow \{g^n x : n \in \mathbb{Z}\}$ dense $\Rightarrow 4 \epsilon > 0 \exists \ n \in \mathbb{Z} : \text{distance between points of } \ell < \epsilon$.

  $\Rightarrow \ell$ dense in $T_2$.  

[Figure showing the winding line $\ell$ and its behavior under translation $f_t$.]
4. Powers of 2

Consider the first digits of the numbers $2^n$ ($n = 0, 1, 2, \ldots$): 1, 2, 4, 8, 1, 3, 6, \ldots

a) Does the number 7 appear in this series? If so, how often?

b) Let $N_7(n)$ and $N_8(n)$ be the number of times 7 and 8, respectively, appear in the first n numbers. Find the relative frequency of occurrence, $\lim_{n \to \infty} \frac{N_7(n)}{N_8(n)}$.

(6 points)

Solution:

first digit of $2^n = \text{integer part of } (10^n \log 2 \pmod{1})$ 
⇒ problem equivalent to mapping of one-circle $S_1$ with circumference 1 onto itself 
Define $g : S_1 \to S_1$ as rotation through angle $2\pi \log 2$

a) $\log 2$ is not rational ⇒ $g^n(0)$ is dense in $S_1$ by problem 2.
⇒ all digits, in particular 7, appear infinitely often

b) 8 appears if $g^n$ maps 0 into $I_8 := [\log 8, \log 9[$
7 8 appears if $g^n$ maps 0 into $I_8 := [\log 7, \log 8[$
measure $(I_8) = \log 9 - \log 8 = \log(9/8)$
measure $(I_7) = \log 8 - \log 7 = \log(8/7)$
⇒ 7 appears $[\log(8/7)/\log(9/8)] = 1.134 \ldots$ times as often as 8

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