1. **Phase flow**

Consider a point mass in $d = 1$ subject to a constant force f, so that the equation of motion is

$$\ddot{q}(t) = f/m \quad .$$

a) Determine the phase flow. Does Liouville's theorem hold?

b) Does Poicaré's theorem hold? Justify your answer.

(4 points)

Solution:

$$\ddot{q} = f/m \rightsquigarrow \dot{q} = \dot{q}_0 + (f/m)t \rightsquigarrow p = m\dot{q} = p_0 + ft$$

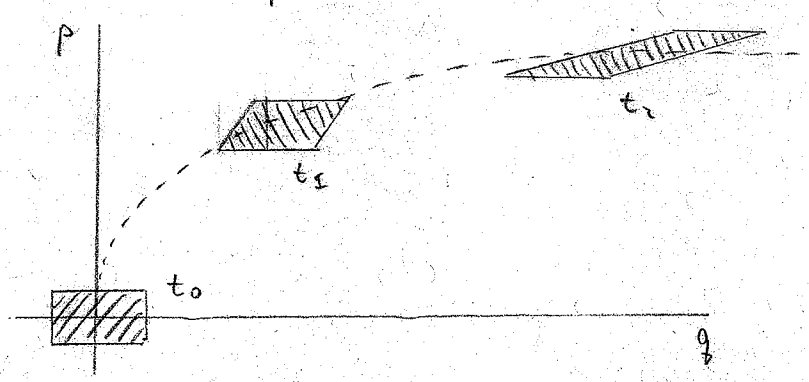
$$\rightsquigarrow \boxed{\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} (f/2m)t^2 \\ ft \end{pmatrix} + \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_0 \\ p_0 \end{pmatrix}}$$

1

a) $z(t) = \begin{pmatrix} (f/2m)t^2 \\ ft \end{pmatrix} + D(t)z_0$ with $D(t) = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix}$

translation along parabola

distortion: Δq increases linearly with t, Δp is constant



$\det D(t) = 1$

$\rightsquigarrow$ Liouville's theorem holds

1 1

b) Poincaré's theorem obviously does not hold

reason: This is not a mapping of a bounded region onto itself!

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2. Circle tranlations

Let S_1 be the 1-sphere (a.k.a. circle), and let g_α be a rotation of S_1 about its center through an angle α. Show that

a) There exists an $n \in \mathcal{Z}$ so that any point $x \in S_1$ returns to its original position after n rotations if and only if $\alpha/2\pi \in \mathcal{Q}$.

b) If $\alpha/2\pi \notin \mathcal{Q}$, then the set of points $\{g^k x;\ k \in \mathcal{Z}\}$ is dense in S_1.

(5 points)

notation: $\mathcal{Z}$ = integers, $\mathcal{Q}$ = rationals.

Solution:

$S_1 = \{ x;\ x \in \mathbb{R} \pmod{2\pi} \}$

$g_\alpha : S_1 \to S_1$ with $g_\alpha x = x + \alpha \pmod{2\pi}$

For $x \in S_1$, define $N_x := \{ g_\alpha^n x;\ n \in \mathbb{Z} \}$

a) Let $\alpha/2\pi \in \mathbb{Q} \to \exists\, m, n \in \mathbb{Z} :\ \alpha/2\pi = m/n$

$\to g_\alpha x = x + \alpha \pmod{2\pi} = x + 2\pi m/n \pmod{2\pi}$

(1) $\to \underline{g_\alpha^n x} = x + 2\pi m \pmod{2\pi} = \underline{x}$ exact return

Let $g_\alpha^n x = x$ for some $n \in \mathbb{Z}$

(1) $\to \exists\, m \in \mathbb{Z} :\ x + n\alpha = x + 2\pi m \to \alpha = 2\pi m/n \to \underline{\alpha \in \mathbb{Q}}$

b) Let $\alpha/2\pi \notin \mathbb{Q}$. a) $\to N_x$ is countable, and has pairwise different elements

Let $x \in S_1$, let $U \equiv U_{\varepsilon/3}(x)$ be $\varepsilon/3$-neighborhood of x

(1) P-theorem $\to \exists\, n \geq 1$ and a $V \subset U :\ g^n V \subset U$

Let $x_0 \in V$. $\to 0 < \underline{d} := |g^n x - x| = |g^n(x - x_0) + (g^n x_0 - x_0) + (x_0 - x)|$

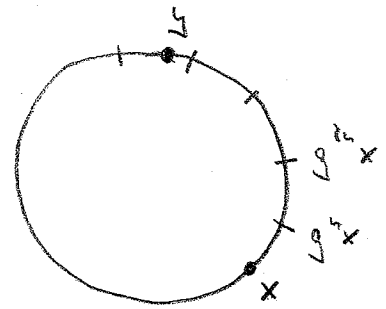
$$\leq \underbrace{|g^n(x-x_0)|}_{=|x-x_0|<\varepsilon/3 \text{ by volume conservation}} + \underbrace{|g^n x_0 - x_0|}_{<\varepsilon/3 \text{ by P-theorem}} + \underbrace{|x_0 - x|}_{<\varepsilon/3 \text{ by construction}} < 3 \times \varepsilon/3 = \underline{\varepsilon}$$

(1)

Let $y \in S_1$ arbitrary, fixed with $|y - x| =: \tilde{d}$

$\rightsquigarrow \exists\, k \in \mathbb{N} : \quad k-1 \leq \tilde{d}/d < k$

$\rightsquigarrow |g^{kn} x - y| \leq d < \varepsilon$



1 Thus $\forall \varepsilon \rightsquigarrow$ N_x dense in S^1

3. Torus translations

Let $T_2 = S_1 \times S_1$ be the 2-torus (i.e., the ordinary doughnut, or the set of points $x = (\phi_1, \phi_2)$, $\phi_i \in \mathcal{R}$ $(i = 1, 2)$, and $(\phi_1, \phi_2) = (\psi_1, \psi_2)$ if and only if $(\phi_i - \psi_i)/2\pi \in \mathcal{Z}$ $(i = 1, 2)$.) Let *torus translations* be defined by

$$f_t(\phi_1, \phi_2) = (\phi_1 + c_1 t, \phi_2 + c_2 t), \quad t \in \mathcal{R}, \quad 0 \neq c_i \in \mathcal{R}, \quad (i = 1, 2) \quad .$$

The *winding line* ℓ through a point $x_0 \in T_2$ is defined as the set

$$\ell = \{x \in T_2;\ x = f_t(x_0),\ t \in \mathcal{R}\} \quad .$$

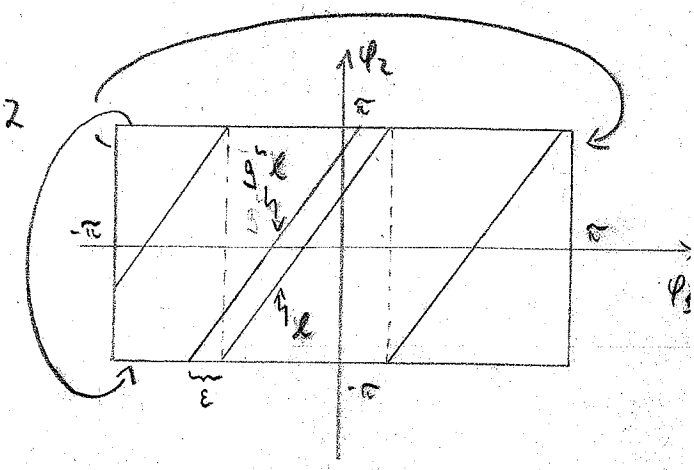
Show that

a) ℓ is closed if and only if $c_1/c_2 \in \mathcal{Q}$.

b) If ℓ is not closed, then ℓ is dense in T_2.

(4 points)

notation: $\mathcal{R}$ = reals, $\mathcal{Q}$ = rationals, $\mathcal{Z}$ = integers.



Solution: by reduction to problem 2

Define angle $\alpha = 2\pi/c_2$

and mapping $f_\alpha(\varphi_1, \varphi_2) = (\varphi_1 + \alpha, \varphi_2)$

In particular, f_α maps

$S_1 = \{(x, 0)\}$ onto $\{(x + \alpha, 0)\}$

(2) Now define angle $\beta = 2\pi c_1/c_2$, and $f_\beta : S_1 \to S_1$

a) Let ℓ be closed $\Leftrightarrow \exists\, n \in \mathbb{Z} : g^n x = x \Leftrightarrow \exists\, m \in \mathbb{Z} : x + n\beta = x + 2\pi m$

$\Leftrightarrow \beta/2\pi = m/n \in \mathbb{Q} \Leftrightarrow c_1/c_2 \in \mathbb{Q}$

(1)

b) Let ℓ be not closed $\to c_1/c_2 \notin \mathbb{Q} \overset{\text{problem 2}}{\to} \{g^n x;\ n \in \mathbb{Z}\}$ dense in S

$\to \forall \varepsilon > 0\ \exists\, n \in \mathbb{Z}$: distance between points of $\ell < \varepsilon$

$\to \ell$ dense in T_2.

(1)

4. **Powers of 2**

Consider the first digits of the numbers 2^n $(n = 0, 1, 2, \ldots)$: $1, 2, 4, 8, 1, 3, 6 \ldots$

a) Does the number 7 appear in this series? If so, how often?

b) Let $N_7(n)$ and $N_8(n)$ be the number of times 7 and 8, respectively, appear in the first n numbers. Find the relative frequency of occurrence, $\lim_{n\to\infty} N_7(n)/N_8(n)$.

(6 points)

Solution:

first digit of 2^n = integer part of $\left(10^{n \log 2 \ (\mathrm{mod} 1)}\right)$ 1pt

$\Rightarrow$ problem equivalent to mapping of one-circle S_1 with circumference 1 onto itself 1pt

Define $g: S_1 \to S_1$ as rotation through angle $2\pi \log 2$ 1pt

a) $\log 2$ is not rational $\Rightarrow g^n(0)$ is dense in S_1 by problem 2).

$\Rightarrow$ all digits, in particular 7, appear infinitely often 1pt

b) 8 appears if g^n maps 0 into $I_8 := [\log 8, \log 9[$

7 8 appears if g^n maps 0 into $I_8 := [\log 7, \log 8[$ 1pt

measure $(I_8) = \log 9 - \log 8 = \log(9/8)$

measure $(I_7) = \log 8 - \log 7 = \log(8/7)$

$\Rightarrow$ 7 appears $[\log(8/7)/\log(9/8)] = 1.134\ldots$ times as often as 8 1pt