PH 614 STATISTICAL MECHANICS Spring 2022

Problem Assignment # 5 4/28/2022

due 5/05/2022

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13. Paramagnets

Consider a system of N noninteracting spins that can point either up of down in an external magnetic field B. Each up-spin contributes an energy $-\mu B$, and each down-spin contributes an energy μB, so that the total energy of a state with n_+ up-spins and n_- down-spins is

$$E_{n_+ n_-} = -(n_+ - n_-)\mu B \quad ,$$

where μ is the spin's magnetic moment. Let the system be in contact with a heat bath of temperature T.

a) Calculate the system's partition function.

b) Calculate and discuss the internal energy, and the specific heat.

c) Calculate the magnetization M from the net average number of up-spins, $\langle n_+ - n_- \rangle$, and also from the free energy as in ch. 4 §4.6, and show that both arguments give the same answer.

d) Calculate the zero-field magnetic susceptibility $\chi = (\partial M/\partial B)_{B=0}$.

(7 points)

14. Balances

A spring balance consists of a harmonic spring (spring constant α) in an environment kept at temperature T. The gravitational acceleration is g.

a) An object of mass m is suspended from the spring. What is the resulting mean elongation $< x >$ of the spring?

b) What is the rms fluctuation, $\langle (x - < x >)^2 \rangle^{1/2}$, of the elongation due to thermal fluctuations?

c) Suppose you have measured a mass that is so small that the thermal fluctuations just barely let you do the measurement. Now you want to measure a mass that is about ten times smaller. How much do you have to reduce the temperature?

(5 points)

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15. Velocity averages

A classical gas (molecular mass m) is in thermal equilibrium at temperature T. Let $\vec{v} = (v_x, v_y, v_z)$ be the velocity of a gas molecule, and $v = |\vec{v}|$ its speed.

a) Find the following averages:

$$\langle v_x \rangle \quad , \quad \langle v_x^2 \rangle \quad , \quad \langle v^2 v_x \rangle \quad , \quad \langle v_x^2 v_y^2 \rangle \quad .$$

b) Find $\langle 1/v \rangle$. Compare this with $1/\langle v \rangle$.

c) Find the mean number of molecules per unit volume whose kinetic energy lies in the interval $[\epsilon, \epsilon + d\epsilon]$.

(5 points)

16. Planetary atmospheres

The moon has a mass $M = 7.35 \times 10^{25}$ g, a radius $R = 1.74 \times 10^8$ cm, and a daytime surface temperature $T = 400$ K.

a) What is the escape velocity of a mass m from the moon (i.e. the minimum velocity necessary to escape the moon's gravitational field?

b) Assuming a lunar atmosphere consisting of H_2, N_2, and CO_2, respectively, in equilibrium at the daytime surface temperature, what would the mean velocities of the respective molecules be, and how do they compare with the escape velocity?

c) Suppose the moon originally had an N_2 atmosphere. What fraction of the molecules were fast enough to escape? Suppose it takes a time τ for these molecules to escape, and for the remaining atmospheric molecules to regain equilibrium. How long does it take (in units of τ) for the density of the atmosphere to drop to 1% of its original value?

hint: Do the relevant integral numerically.

d) Estimate τ by assuming a wind speed $c \approx 100$ m/s, and a height of the atmosphere $h \approx 10$ km. How long would it have taken (in years) for the moon to loose that atmosphere? How does this compare to the age of the moon? What do you conclude from your result in conjunction with the recent finding that there is water ice on the moon (albeit in well-sheltered spots near the poles, where the daytime temperature is quite a bit lower than 400 K)?

e) The corresponding values for the earth are $M = 5.98 \times 10^{27}$ g, $R = 6.38 \times 10^8$ cm, and $T = 300$ K. Are you worried about the earth losing its atmosphere?

hint: Estimate a lower limit for the lifetime of the earth's N_2 atmosphere.

(9 points)

13.) $E_{n_+ n_-} = -(n_+ - n_-)\mu B$

$n_+, n_- = 0, 1, \dots, N \quad , \quad n_+ + n_- = N$

a) There are $\binom{N}{n_+}$ ways to pick n_+ up-spins, leading to $\binom{N}{n_+}$ degenerate microstates for a given n_+

(1) $$\rightarrow \underline{Z} = \sum_{n_+, n_-} \binom{N}{n_+} \delta_{n_+ + n_-, N} \, e^{\beta\mu B(n_+ - n_-)}$$

$$= \sum_{n=0}^{N} \binom{N}{n} e^{\beta\mu B(2n - N)}$$

$$= e^{-\beta N \mu B} \sum_{n=0}^{N} \binom{N}{n} \left(e^{2\beta\mu B}\right)^n$$

binomial thm $$= e^{-\beta N \mu B} \left(1 + e^{2\beta\mu B}\right)^N = \left(e^{-\beta\mu B} + e^{\beta\mu B}\right)^N$$

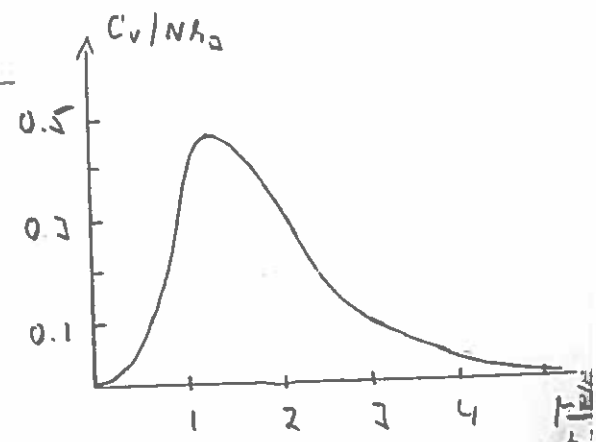
(1) $$= \underline{\underline{\left[2\cosh(\beta\mu B)\right]^N}}$$

b) $$\underline{\underline{U}} = -\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} N \ln\left(2\cosh(\beta\mu B)\right) = -N\mu B \frac{\sinh(\beta\mu B)}{\cosh \beta\mu B}$$

$$= \underline{\underline{-N\mu B \tanh(\beta\mu B)}}$$

$$\underline{\underline{C_V}} = \frac{\partial U}{\partial T} = -N(\mu B)^2 \frac{-1}{k_B T^2} \frac{1}{\cosh^2(\beta\mu B)} = \frac{N(\mu B)^2}{k_B T^2} \frac{1}{\cosh^2(\mu B/k_B T)}$$

(1) $$= \underline{\underline{N k_B \left(\frac{\mu B}{k_B T}\right)^2 \frac{1}{\cosh^2(\mu B/k_B T)}}}$$



C_V has a characteristic peak structure ('Schottky anomaly')

(1)

c) I can calculate the magnetization as

$$\underline{m} = \mu \langle n_+ - n_- \rangle = \mu \frac{1}{Z} \sum_{n_+, n_-} (n_+ - n_-) \binom{N}{n_+} \delta_{n_+ + n_-, N} \, e^{\beta \mu B (n_+ - n_-)}$$

$$= \frac{1}{\beta} \frac{1}{Z} \frac{\partial}{\partial B} \sum_{n_+, n_-} \binom{N}{n_+} \delta_{n_+ + n_-, N} \, e^{\beta \mu B (n_+ - n_-)}$$

$$= \frac{1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial B} = \frac{1}{\beta} \frac{\partial \log Z}{\partial B} = \frac{1}{\beta} \frac{\partial}{\partial B} N \log \left[2 \cosh (\beta \mu B) \right]$$

(1) $$= \underline{\underline{N \mu \tanh (\beta \mu B)}}$$

Alternatively, I can use thermodynamics: cf § 4.6 ⇝

$$\underline{m} = \frac{-\partial F}{\partial B} = -\frac{\partial}{\partial B} \left(-k_B T \log Z \right) = \underline{\frac{1}{\beta} \frac{\partial \log Z}{\partial B}}$$ which gives the same answer.

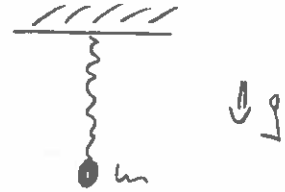
<u>remark</u>: The way the problem is formulated we fix B and let m adjust ⇝ Our Z determines what we called $\tilde{F}(B)$ in cf § 4.6, as opposed to $F(m) = \tilde{F} + Bm$.

(1)

d) $$\underline{\underline{\chi}} = \left(\frac{\partial m}{\partial B} \right)_{B=0} = \frac{N \mu^2 \beta}{\cosh^2 (\beta \mu B)} \Big|_{B=0} = N \mu^2 \beta = \underline{\underline{\frac{N \mu^2}{k_B T}}}$$ <u>Curie law</u>

(1)

14.) Potential energy: $U = mgx + \frac{\alpha}{2}x^2$



a) Equilibrium position:

(1) $$0 \overset{!}{=} \left.\frac{dU}{dx}\right|_{\langle x\rangle} = mg + \alpha\langle x\rangle \rightarrow \underline{\langle x\rangle = -mg/\alpha}$$

b) $$U = mg\left(\langle x\rangle + \delta x\right) + \frac{\alpha}{2}\left(\langle x\rangle^2 + 2\langle x\rangle\delta x + (\delta x)^2\right)$$

(1) $$= \text{const} + \frac{\alpha}{2}(\delta x)^2$$

$$\rightarrow \underline{\langle(\delta x)^2\rangle} = \int_{-\infty}^{\infty} d(\delta x)\,(\delta x)^2\, e^{-\Delta\alpha(\delta x)^2/2} \Big/ \int_{-\infty}^{\infty} d(\delta x)\, e^{-\Delta\alpha(\delta x)^2/2}$$

(1) $$= \frac{2}{\Delta\alpha}\int dx\, x^2 e^{-x^2} \Big/ \int dx\, e^{-x^2} = \frac{1}{\Delta\alpha} = \underline{k_B T/\alpha}$$

c) The measurement becomes impossible if the fluctuations become comparable to the average:

(1) $$mg/\alpha = |\langle x\rangle| \simeq \langle(\delta x)^2\rangle^{1/2} = \sqrt{k_B T/\alpha}$$

$\rightarrow$ The lower bound for measurable masses is

$$\underline{m \gtrsim \frac{1}{g}\sqrt{\alpha k_B T}}$$

(1) $\rightarrow$ In order to measure a mass $m/10$ one must go to a temperature $T/100$.

15.) Maxwell-Boltzmann:

$$\boxed{\rho(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-mv^2/2k_BT}}$$

$$\boxed{\rho(v) = 4\pi\left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 e^{-mv^2/2k_BT}}$$

a) $\underline{\underline{\langle v_x \rangle}} = \int dv_x\, v_x \int dv_y\, dv_z\, \rho(\vec{v}) = \underline{\underline{0}}$ by symmetry

$\underline{\underline{\langle v_x^2 \rangle}} = \frac{1}{3}\langle v_x^2 + v_y^2 + v_z^2 \rangle$ by symmetry

(1) $= \frac{1}{3}\langle v^2 \rangle \overset{\S 14}{=} \frac{1}{3}\, 3 k_BT/m = \underline{\underline{k_BT/m}}$

$\underline{\underline{\langle v^2 v_x \rangle}} = \underline{\underline{0}}$ by symmetry

(1) $\underline{\underline{\langle v_x^2 v_y^2 \rangle}} = \langle v_x^2 \rangle \langle v_y^2 \rangle = \underline{\underline{(k_BT/m)^2}}$

b) $\underline{\underline{\langle 1/v \rangle}} = \int_0^\infty dv\, \frac{1}{v}\rho(v) = 4\pi\left(\frac{m}{2\pi k_BT}\right)^{3/2}\int_0^\infty dv\, v\, e^{-mv^2/2k_BT}$

$= \cancel{4\pi}\, \frac{2k_BT}{\cancel{m}}\, \frac{\cancel{m}}{\cancel{2\pi k_BT}}\left(\frac{m}{2\pi k_BT}\right)^{1/2} \frac{1}{\cancel{2}} \underbrace{\int_0^\infty dx\, e^{-x}}_{=1} = \underline{\underline{\sqrt{2m/\pi k_BT}}}$

whereas

(1) $\underline{\underline{\frac{1}{\langle v \rangle} = \sqrt{\pi m/8k_BT} < \langle 1/v \rangle}}$

c) kinetic energy: $\varepsilon = mv^2/2 \rightarrow d\varepsilon = mv\,dv$

$\rightarrow \rho(v)\,dv = \rho(\varepsilon)\,d\varepsilon = \rho(\varepsilon)\,mv\,dv$

$\rightarrow \underline{\underline{\rho(\varepsilon)}} = \frac{1}{mv}\rho(v)\Big|_{v=\sqrt{2\varepsilon/m}} = \frac{4\pi}{\cancel{m}}\left(\frac{m}{2\pi k_BT}\right)^{3/2}\left(\frac{2\varepsilon}{\cancel{m}}\right)^{1/2} e^{-\varepsilon/k_BT}$

(1) $= \frac{2\pi}{(\pi k_BT)^{3/2}}\sqrt{\varepsilon}\, e^{-\varepsilon/k_BT}$

number density: $n = N/V$ molecules per volume

$\Rightarrow$ $\underline{\underline{n\, g(\varepsilon)\, d\varepsilon}}$ is the number of molecules per unit volume with kinetic energy $\in [\varepsilon, \varepsilon + d\varepsilon]$

(1)

16) a) Escape velocity: $\frac{m}{2} v_e^2 = G \frac{mM}{R}$, $G = 6.67 \times 10^{-8}$ cgs

(1) $\rightsquigarrow$ $\underline{\underline{v_e}} = \sqrt{2GM/R} = \underline{\underline{2.37 \times 10^5 \text{ m/s}}}$

b) $\langle v \rangle = \left(\frac{8kT}{\pi m} \right)^{1/2} = \frac{3.75 \times 10^{-7}}{\sqrt{m}} \frac{m}{s}$ with m measured in g

H_2 : $m = 3.34 \times 10^{-24}$ g $\rightsquigarrow$ $\langle v \rangle = 2.05 \times 10^5$ m/s $\simeq v_e$

N_2 : $m = 4.68 \times 10^{-23}$ g $\rightsquigarrow$ $\langle v \rangle = 5.48 \times 10^4$ m/s $\simeq v_e/4$

CO_2 : $m = 7.35 \times 10^{-23}$ g $\rightsquigarrow$ $\langle v \rangle = 4.37 \times 10^4$ m/s $\simeq v_e/6$

c) fraction of molecules just enough to escape

$$\frac{N_e}{N} = \int_{v_e}^{\infty} dv\, g(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{v_e}^{\infty} dv\, v^2 e^{-\frac{m}{2k_B T} v^2}$$

$$= 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \left(\frac{2k_B T}{m}\right)^{3/2} \int_{v_e/\sqrt{m/2k_B T}}^{\infty} dx\, x^2 e^{-x^2}$$

$$= \frac{2}{\sqrt{\pi}} \int_{\frac{m}{2} v_e^2/k_B T}^{\infty} dx\, x^{1/2} e^{-x} =: f\left(\tfrac{m}{2} v_e^2/k_B T\right)$$

(1)

with $f(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} dy\, y^{1/2} e^{-y}$

$$\frac{m}{2} v_e^2/k_B T = \frac{4.68 \times 10^{-23} (2.37 \times 10^5)^2}{2 \times 1.38 \times 10^{-16} \times 4{,}000} \simeq 24$$

$\rightarrow \frac{N_e}{N} \simeq 2 \times 10^{-10}$ (numerically)

(1)

$\rightarrow$ $\frac{N_e}{N}$ of the atmosphere escapes within a time τ

$\rightarrow$ after a time $n\tau$, the density is $\left(1 - \frac{N_e}{N}\right)^n$ of the original value.

$$(1 - 2 \times 10^{-10})^n \overset{!}{=} \frac{1}{100} \rightarrow n \log(1 - 2 \times 10^{-10}) \simeq -n \times 2 \cdot 10^{-10} = -\log 100$$

$$\rightarrow n = \frac{+\log 100}{2 \times 10^{-10}} \simeq 2 \times 10^{10}$$

(1)

$\rightarrow$ The atmosphere is essentially gone after a time $t \sim 2 \times 10^{10}\, \tau$

d) $\tau = h/c \simeq \dfrac{10^4\ \mathrm{m}}{10^2\ \mathrm{m/s}} = 10^2\ \mathrm{s}$

$\rightarrow$ $\underline{\underline{t}} \simeq 2\times10^{10}\times10^2\ \mathrm{s} = 2\times10^{12}\ \mathrm{s} \simeq \underline{\underline{10^5\ \mathrm{yrs}}}$

By contrast, the age of the moon is about the same as the age of the earth, or $\underline{\underline{t_{moon} \simeq 5\times10^9\ \mathrm{yrs}}}$

(1)

(1) The observed water must be replenished (e.g. by comets)

e) For the earth, $\underline{v_e = 2.14\times10^6\ \mathrm{m/s}}$

$\rightarrow$ $\frac{m}{2}v_e^2/k_BT \simeq 108$

$$f(108) = \frac{2}{\sqrt{\pi}}\int_{108}^{\infty} dx\,\sqrt{x}\,e^{-x} < \frac{2}{\sqrt{\pi}}\int_{108}^{\infty} dx\,x\,e^{-x} \simeq \frac{2}{\sqrt{\pi}}\,108\times e^{-108}$$

$$\rightarrow \underline{\underline{t/\tau}} > \frac{\ln 100}{100\times e^{-100}} \simeq 5\times10^{-2}\,e^{100} \simeq 5\times10^{-2}\times2\times10^{43}$$

(1) $\simeq \underline{\underline{10^{42}}}$

$\rightarrow$ $\underline{\underline{t}} > 10^{44}\ \mathrm{s} \simeq 3\times10^{36}\ \mathrm{yrs} \gg$ age of the universe ($\simeq 10^{10}$ yrs)

(1) $\rightarrow$ the earth's atmosphere is safe!