13. Paramagnets

Consider a system of $N$ noninteracting spins that can point either up or down in an external magnetic field $B$. Each up-spin contributes an energy $-\mu B$, and each down-spin contributes an energy $\mu B$, so that the total energy of a state with $n_+$ up-spins and $n_-$ down-spins is

$$E_{n_+, n_-} = -(n_+ - n_-)\mu B,$$

where $\mu$ is the spin’s magnetic moment. Let the system be in contact with a heat bath of temperature $T$.

a) Calculate the system’s partition function.

b) Calculate and discuss the internal energy, and the specific heat.

c) Calculate the magnetization $M$ from the net average number of up-spins, $\langle n_+ - n_- \rangle$, and also from the free energy as in ch. 4 §4.6, and show that both arguments give the same answer.

d) Calculate the zero-field magnetic susceptibility $\chi = (\partial M/\partial B)_{B=0}$.

(7 points)

14. Balances

A spring balance consists of a harmonic spring (spring constant $\alpha$) in an environment kept at temperature $T$. The gravitational acceleration is $g$.

a) An object of mass $m$ is suspended from the spring. What is the resulting mean elongation $<x>$ of the spring?

b) What is the rms fluctuation, $\langle (x - <x>)^2 \rangle^{1/2}$, of the elongation due to thermal fluctuations?

c) Suppose you have measured a mass that is so small that the thermal fluctuations just barely let you do the measurement. Now you want to measure a mass that is about ten times smaller. How much do you have to reduce the temperature?

(5 points)
15. Velocity averages

A classical gas (molecular mass \( m \)) is in thermal equilibrium at temperature \( T \). Let \( \vec{v} = (v_x, v_y, v_z) \) be the velocity of a gas molecule, and \( v = |\vec{v}| \) its speed.

a) Find the following averages:
\[
\langle v_x \rangle, \quad \langle v_x^2 \rangle, \quad \langle v^2 v_x \rangle, \quad \langle v_x^2 v_y^2 \rangle.
\]

b) Find \( \langle 1/v \rangle \). Compare this with \( 1/\langle v \rangle \).

c) Find the mean number of molecules per unit volume whose kinetic energy lies in the interval \( [\epsilon, \epsilon + d\epsilon] \).

(5 points)

16. Planetary atmospheres

The moon has a mass \( M = 7.35 \times 10^{25} \) g, a radius \( R = 1.74 \times 10^8 \) cm, and a daytime surface temperature \( T = 400 \) K.

a) What is the escape velocity of a mass \( m \) from the moon (i.e. the minimum velocity necessary to escape the moon’s gravitational field)?

b) Assuming a lunar atmosphere consisting of \( \text{H}_2, \text{N}_2, \) and \( \text{CO}_2 \), respectively, in equilibrium at the daytime surface temperature, what would the mean velocities of the respective molecules be, and how do they compare with the escape velocity?

c) Suppose the moon originally had an \( \text{N}_2 \) atmosphere. What fraction of the molecules were fast enough to escape? Suppose it takes a time \( \tau \) for these molecules to escape, and for the remaining atmospheric molecules to regain equilibrium. How long does it take (in units of \( \tau \)) for the density of the atmosphere to drop to 1% of its original value?

*hint:* Do the relevant integral numerically.

d) Estimate \( \tau \) by assuming a wind speed \( c \approx 100 \) m/s, and a height of the atmosphere \( h \approx 10 \) km. How long would it have taken (in years) for the moon to lose that atmosphere? How does this compare to the age of the moon? What do you conclude from your result in conjunction with the recent finding that there is water ice on the moon (albeit in well-sheltered spots near the poles, where the daytime temperature is quite a bit lower than 400 K)?

e) The corresponding values for the earth are \( M = 5.98 \times 10^{27} \) g, \( R = 6.38 \times 10^8 \) cm, and \( T = 300 \) K. Are you worried about the earth losing its atmosphere?

*hint:* Estimate a lower limit for the lifetime of the earth’s \( \text{N}_2 \) atmosphere.

(9 points)
The image contains a mathematical diagram and equations, but the content is not legible due to the quality of the scan. The text appears to be related to chemical or physical processes, possibly involving concentration or reaction rates, but the specific details are not discernible.
(c) & can calculate the momentum as
\[ \Pi = \mu \langle \Pi_{+} - \Pi_{-} \rangle = \mu \langle \frac{1}{\lambda} \sum_{n_+ \neq n_-} \delta_{n_+, n_-} \exp \left( \frac{\lambda}{2} \right) \rangle \]
\[ = \frac{1}{\lambda} \sum_{n_+ \neq n_-} \frac{1}{\lambda} \exp \left( \frac{\lambda}{2} \right) \delta_{n_+, n_-} \exp \left( \frac{\lambda}{2} \right) \]
\[ = \frac{1}{\lambda} \frac{d}{d\lambda} \sum_{n_+ \neq n_-} \frac{1}{\lambda} \exp \left( \frac{\lambda}{2} \right) \delta_{n_+, n_-} \exp \left( \frac{\lambda}{2} \right) \]
\[ = \frac{1}{\lambda} \frac{d}{d\lambda} N \log \left( \frac{2}{\cosh (\lambda)} \right) \]
\[ = N \mu \tanh (\lambda \mu) \]

Attaching, & can use thermodynamics: \( d I \approx \frac{d}{dt} \sum_{j \neq 0} \)
\[ d = -\frac{\partial \mathcal{E}}{\partial T} = -\frac{\partial}{\partial T} \left( -\lambda_0 \cot \theta T \right) = -\frac{1}{\lambda} \frac{d}{d\lambda} \lambda_0 \cot \theta \]

Remark: The way the problem is framed is crucial in this case. Let \( H \) be adjusted to \( 0 \) or determine \( \lambda_0 \) to be called \( \mathcal{E}(\theta) \) & \( \lambda \approx 4.6 \), as opposed to \( \mathcal{E}(\lambda) = \mathcal{E} + \lambda \mathcal{H} \).

d) \[ \chi = \left( \frac{\partial }{\partial \lambda} \right)_{\lambda=0} = \frac{N \mu^2 \lambda}{\omega \cosh^2 (\lambda \mu)} \bigg|_{\lambda=0} = N \mu^2 \lambda = \frac{N \mu^2}{\lambda^2} \quad \text{Limit law} \]
14. Potential energy: \( U = mgx + \frac{k}{2}x^2 \)

a) Equilibrium position:
\[
0 = \frac{dU}{dx} \bigg|_{\langle x \rangle} = mg + k\langle x \rangle \Rightarrow \langle x \rangle = -\frac{mg}{k}
\]

b) \( U = mg(\langle x \rangle; \delta x) + \frac{k}{2} \left( \langle x \rangle^2 + 2\langle x \rangle \delta x + (\delta x)^2 \right) \)
\[
= mgx + \frac{k}{2} (\delta x)^2
\]
\[
\Rightarrow \langle (\delta x)^2 \rangle = \int_{-\infty}^{\infty} d(\delta x) (\delta x)^2 e^{-\lambda x (\delta x)^2/2} \int_{-\infty}^{\infty} d(\delta x) e^{-\lambda x (\delta x)^2/2}
\]
\[
= \frac{2}{\lambda x} \int_{-\infty}^{\infty} dx x^2 e^{-x^2} / \int_{-\infty}^{\infty} dx e^{-x^2} = \frac{1}{\lambda x} = \frac{\hbar}{2kT}
\]

c) The moment comes at a point if the fluctuations become unresolvable to the average:
\[
m\frac{\hbar}{k} = |\langle x \rangle| = \langle (\delta x)^2 \rangle^{1/2} = \sqrt{\frac{\hbar}{2kT}}
\]
\[
\Rightarrow \text{The horizon for measurable mean is}
\]
\[
m \geq \frac{1}{2} \sqrt{\frac{kT\hbar}{2}}
\]
\[
\Rightarrow \text{In order to measure a mass of \(m/10\) one must go to}
\]
\[
\text{the potential } T/100.
\]
\[ g(\nu) = \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} e^{-\nu^2/2\hbar^2} \]
\[ g(\nu) = 4\pi \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} \nu^2 e^{-\nu^2/2\hbar^2} \]

a) \[ \langle v_x \rangle = \int dv_x v_x f(\nu) = 0 \] 
\[ \langle v_x' \rangle = \frac{1}{2} \langle v_x^2 + v_y^2 + v_z^2 \rangle = \frac{1}{2} \langle v^2 \rangle = \frac{1}{2} \left( \frac{k_B T}{m} \right) \nu^2 \]
\[ \langle v_x' v_x \rangle = 0 \] 
\[ \langle v_x' v_y' \rangle = \langle v_x' \rangle \langle v_y' \rangle = \left( \frac{k_B T}{m} \nu \right) \]

b) \[ \langle \nu v \rangle = \int_0^\infty dv \nu f(\nu) = 4\pi \left( \frac{m}{2\pi \hbar^2} \right)^{3/2} \int_0^\infty dv \nu e^{-\nu^2/2\hbar^2} \]
\[ = \frac{4\pi}{2\hbar^2} \left( \frac{m}{2\pi \hbar^2} \right)^{1/2} \left( \frac{m}{2\pi \hbar^2} \right)^{1/2} \int_0^\infty dv e^{-v^2} = \frac{2\pi}{\hbar^2} \left( \frac{m}{2\pi \hbar^2} \right)^{1/2} \]

where \[ \frac{1}{\langle \nu \rangle} = \frac{1}{\sqrt{8\pi \hbar^2}} < \langle \nu \rangle \]

c) kinetic energy: \[ \epsilon = \frac{\nu^2}{2} \rightarrow \delta\epsilon = \nu d\nu \]
\[ \rightarrow \frac{d\nu}{\nu} = \int d\epsilon = \int (\delta\epsilon) d\nu \]
\[ \rightarrow \frac{g(\nu)}{\nu} = \left. \frac{1}{\nu} g(\nu) \right|_{\nu = 12\hbar^2/m} = \frac{k_B T}{m} \left( \frac{2\pi \hbar^2}{m} \right)^{1/2} \frac{\nu}{2} e^{-\nu^2/2\hbar^2} \]
\[ = \frac{2\pi}{\sqrt{2\pi}} \int_0^\infty e^{-x^2} dx \]
molar density: $n = N/V$ molecules per volume

$\nu g(\varepsilon)d\varepsilon$ is the number of molecules per unit volume with kinetic energy $\nu g(\varepsilon)$ in $[\varepsilon, \varepsilon + d\varepsilon]$. 

(1)
16. a) Escape velocity: \[ \frac{v_e}{R} = \frac{G m}{R} \]
\[ \Rightarrow v_e = \sqrt{\frac{2GM}{R}} = 2.27 \times 10^5 \text{ m/s} \]

b) \[ \langle v \rangle = \left( \frac{8\pi^2 m}{3} \right)^{1/4} = \frac{2.75 \times 10^{-2}}{\sqrt{m}} \frac{m}{4} \text{ m/s} \]

- \[ N_e : m = 2.34 \times 10^{-25} \text{ kg} \Rightarrow \langle v \rangle = 2.05 \times 10^5 \text{ m/s} \]
- \[ N_2 : m = 4.68 \times 10^{-25} \text{ kg} \Rightarrow \langle v \rangle = 5.48 \times 10^4 \text{ m/s} \]
- \[ C_{60} : m = 7.75 \times 10^{-25} \text{ kg} \Rightarrow \langle v \rangle = 4.37 \times 10^5 \text{ m/s} \]
c) fraction of indians \( n = \frac{N_c}{N} \) is \\
\[
\frac{N_c}{N} = \int_0^\infty \frac{\nu}{\nu_c} e^{-\frac{\nu}{\nu_c}} d\nu = 4\pi \left( \frac{h_c}{2\pi \lambda_c^2} \right)^{3/2} \int_0^\infty \frac{\nu}{\nu_c} \nu e^{\frac{-\nu}{\nu_c}} d\nu
\]
\[
= 4\pi \left( \frac{h_c}{2\pi \lambda_c^2} \right)^{3/2} \left( \frac{2\nu_c}{\nu_e} \right)^{3/2} \int_0^\nu_c \frac{\nu}{\nu_c} \nu e^{-\frac{\nu}{\nu_c}} d\nu
\]
\[
= \frac{2}{\nu_c^2} \int_0^\infty \frac{\nu}{\nu_c} \nu e^{-\frac{\nu}{\nu_c}} = f \left( \frac{\nu_c^2}{\lambda_c^2} \right)
\]

\[ f(x) = \frac{2}{\nu_c^2} \int_0^\infty \frac{\nu}{\nu_c} \nu e^{-\frac{\nu}{\nu_c}} \]

\[
\frac{\nu_c^2}{\lambda_c^2} = \frac{4.62 \times 10^{-12} \times (2.77 \times 10^5)^2}{2 - 1.38 \times 10^{-16} \times 4.10^{-4}} \approx 2.4
\]

\[
\Rightarrow \frac{N_c}{N} = 2 \times 10^{-10} \quad \text{(numerically)}
\]

is \( \frac{N_c}{N} \) of the atmosphere escaped within a time \( t \)

after \( t \), the density is \( (1 - \frac{N_c}{N})^{1/2} \) of the original value.

\[
(1 - 2 \times 10^{-10})^{1/2} \Rightarrow \frac{\ln (1 - 2 \times 10^{-10})}{10} \approx -2 \times 10^{-10} = -\ln 100
\]

\[
\Rightarrow n = \frac{\ln 100}{2 \times 10^{-10}} \approx 2 \times 10^{10}
\]

The atmosphere is unlikely gone after a time \( 2 \times 10^{10} \).
\[ t = \frac{h}{c} = \frac{10^{14} \text{ m}}{10^5 \text{ m/s}} = 10^9 \text{s} \]

\[ \Rightarrow t \approx 2 \times 10^9 \times 10^3 \text{s} = 2 \times 10^{12} \text{s} = 10^5 \text{ yr} \]

By contrast, the age of the moon is about the same as the age of the earth, 1.0 or \( t_{\text{moon}} \approx 5 \times 10^9 \text{ yr} \).

The observed water must be replenished (e.g., by comets).

c) For the earth, \( v_e = 2.14 \times 10^6 \text{ m/s} \)

\[ \Rightarrow \sqrt{\frac{v_e^2}{k_B}} = 108 \]

\[ t(108) = \frac{1}{10^2} \int_{10^2} dx \frac{x e^{-x}}{102} < \frac{1}{10^2} \int_{10^2} dx x e^{-x} = \frac{1}{10^2} \frac{108}{110} e^{-10^2} \]

\[ \Rightarrow t/\tau > \frac{100}{100 e^{100}} = 5 \times 10^{-3} e^{100} \approx 5 \times 10^{-3} \times 2 \times 10^{42} \]

\[ = 10^{42} \]

\[ \Rightarrow t > 10^{42} \text{s} = 3 \times 10^{36} \text{ yr} \Rightarrow \text{age of the universe} \]

\[ (\approx 10^{10} \text{ yr}) \]

\[ \Rightarrow \text{the earth's atmophere is safe!} \]