2.2.1. Planar charge distributions

a) Consider a homogeneously charged infinitesimally thin ring with radius $R$ and total charge $Q$ that is oriented perpendicular to the $z$-axis. Calculate the electric field on the $z$-axis.

b) The same for a homogeneously charged disk with charge density $\sigma$ and radius $R$. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they make sense.

(4 points)

2.2.2. Spherically symmetric charge distributions

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\mathbf{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

*hint:* Make an ansatz for a purely radial field, $\mathbf{E}(\mathbf{x}) = E(r) \hat{e}_r$, and integrate Gauss’s law over a spherical volume.

Explicitly calculate and plot the field $\mathbf{E}(\mathbf{x})$ and the potential $\varphi(\mathbf{x})$ for

b) a homogeneously charged sphere

$$\rho(\mathbf{x}) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases}.$$ 

c) a homogeneously charged spherical shell

$$\rho(\mathbf{x}) = \sigma_0 \delta(r - r_0).$$

(8 points)
2.2.3. **Electrostatics in d dimensions (to be continued later)**

Consider the third Maxwell equation in \(d\) dimensions:

\[
\nabla \cdot \mathbf{E}(\mathbf{x}) = S_d \rho(\mathbf{x})
\]

with the electric field \(\mathbf{E}\) a \(d\)-vector, and \(S_d\) the area of the \((d-1)\)-sphere: \(S_{2n} = 2\pi^n/(n-1)!\) and \(S_{2n+1} = 2^{n+1}n!\pi^n/(2n)!\) for even and odd dimensions, respectively. Define a scalar potential \(\varphi(\mathbf{x})\) in analogy to the 3−\(d\) case, such that

\[
\mathbf{E}(\mathbf{x}) = -\nabla \varphi(\mathbf{x})
\]

and consider Poisson’s equation

\[
\nabla^2 \varphi(\mathbf{x}) = -S_d \rho(\mathbf{x})
\]

**Note:** Here we consider a generalization of electrostatics to \(d\)-dimensional space, NOT a \(d\)-dimensional charge distribution embedded in 3-dimensional space.

a) Show that the Green function \(G_d(\mathbf{x})\) function for Poisson’s equation, i.e., the solution of

\[
\nabla^2 G_d(\mathbf{x}) = -S_d \delta(\mathbf{x})
\]

is given by

\[
G_d(\mathbf{x}) = \frac{1}{d-2} \frac{1}{|\mathbf{x}|^{d-2}}
\]

for all \(d \neq 2\), and by

\[
G_2(\mathbf{x}) = \ln(1/|\mathbf{x}|)
\]

for \(d = 2\).

**Hint:** For \(d = 1\), differentiate directly, using \(d \operatorname{sgn} x/dx = 2 \delta(x)\). For \(d \geq 2\), show that \(G_d(\mathbf{x})\) is a harmonic function for all \(\mathbf{x} \neq 0\), then integrate \(\nabla^2 G_d\) over a hypersphere around the origin and use Gauss’s law.

(4 points)