2.2.1. Lindhard function
Consider the function \( f : \mathbb{C} \rightarrow \mathbb{C} \) (which plays an important role in the theory of many-electron systems) defined by
\[
f(z) = \log \left( \frac{z - 1}{z + 1} \right)
\]
The spectrum \( f'' : \mathbb{R} \rightarrow \mathbb{R} \) and the reactive part \( f' : \mathbb{R} \rightarrow \mathbb{R} \) of \( f \) are defined by
\[
f''(\omega) := \frac{1}{2i} [f(\omega + i0) - f(\omega - i0)] \quad \text{and} \quad f'(\omega) := \frac{1}{2} [f(\omega + i0) + f(\omega - i0)]
\]
where \( f(\omega \pm i0) := \lim_{\epsilon \to 0} f(\omega \pm i\epsilon) \).

a) Show that \( f' \) and \( f'' \) are indeed real-valued functions.
b) Determine \( f'' \) and \( f' \) explicitly, and plot them for \(-3 < \omega < 3\).
c) Show that
\[
\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{f''(\omega)}{\omega - z} = f(z)
\]
(5 points)

2.2.2. Another causal function
The function considered in Problem 2.2.1 is an example of a class of complex functions called causal functions that are important in the theory of many-particle systems. Another member of this class is
\[
g(z) = \sqrt{z^2 - 1} - z
\]
Determine the spectrum and the reactive part of \( g(z) \), and plot them for \(-3 < \omega < 3\).
(3 points)

2.2.3. Proof of the Cauchy-Riemann Theorem
Prove the Cauchy-Riemann theorem from ch.2 §2.2:

a) Let \( f(z) = f'(z',z'') + if''(z',z'') \) be analytic everywhere in \( \Omega \subseteq \mathbb{C} \). Show that the Cauchy-Riemann equations
\[
\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \quad \text{and} \quad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}
\]
hold \( \forall z \in \Omega \).

\textit{hint}: Start with the difference quotient \((f(z) - f(z_0))/(z - z_0)\) and require that its limit for \( z \to z_0 \) exists if \( z_0 \) is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

b) Let the Cauchy-Riemann equations hold in a point \( z_0 \in \Omega \). Show that this implies that \( f \) is analytic in the point \( z_0 \).

\textit{hint}: Consider \( f(z) - f(z_0) \) and expand \( f'(z',z'') \) and \( f''(z',z'') \) in Taylor series about \( z_0 \).
(8 points)

... /over
2.2.4. Exponentials

Consider the exponential function

\[ f(z) = e^z = e^{z' + iz''} \]

a) Show that \( f(z) \) is analytic everywhere in \( \mathbb{C} \).

b) Convince yourself explicitly that the real and imaginary parts of \( f \) obey Laplace’s differential equation.

c) Show that \( df/dz|_z = f(z) \).

d) Show that \( \cos z \) and \( \sin z \), defined by

\[
\cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \quad , \quad \sin z = \frac{1}{2i} (e^{iz} - e^{-iz})
\]

are analytic everywhere in \( \mathbb{C} \), and that

\[
\frac{d}{dz} \cos z = -\sin z \quad , \quad \frac{d}{dz} \sin z = \cos z.
\]

(4 points)