2.4.1. 1-d Fourier transforms

Consider a function \( f \) of one real variable \( x \). Calculate the Fourier transforms \( \hat{f}(k) = \int dx \, e^{-ikx} f(x) \) of the following functions:

a) \( f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \).

b) \( f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \).

c) \( f(x) = e^{-(x/x_0)^2} \).

(3 points)

2.4.2. 3-d Fourier transforms

Consider a function \( f \) of one vector variable \( x \in \mathbb{R}^3 \). The Fourier transform \( \hat{f} \) of \( f \) is defined as

\[
\hat{f}(k) = \int dx \, e^{-ik \cdot x} f(x) .
\]

Calculate the Fourier transforms of the following functions:

a) \[
f(x) = \begin{cases} 1 & \text{for } r < r_0 \quad (r = |x|) \\ 0 & \text{otherwise} \end{cases} .
\]

b) \[
f(x) = 1/r .
\]

*hint:* Consider \( g(x) = \frac{1}{r} e^{-r/r_0} \) and let \( r_0 \to \infty \).

(3 points)
2.4.3. More 1-d Fourier transforms

Consider a function of time \( f(t) \) and define its Fourier transform

\[
\hat{f}(\omega) := \int dt \ e^{i\omega t} f(t)
\]

and its Laplace transform \( F(z) \) as

\[
F(z) = \pm i \int dt \ e^{itz} f_\pm(t) \quad (\pm \text{ for sgn(Im } z) = \pm 1)
\]

with \( z \) a complex frequency and \( f_\pm(t) = \Theta(\pm t) f(t) \). Further define

\[
F''(\omega) = \frac{1}{2i} \left[ F(\omega + i0) - F(\omega - i0) \right], \quad F'(\omega) = \frac{1}{2} \left[ F(\omega + i0) + F(\omega - i0) \right]
\]

Calculate \( F''(\omega) \) and \( F'(\omega) \) for

a) \( f(t) = e^{-|t|/\tau} \)

b) \( f(t) = e^{i\omega_0 t} \)

*hint:* \( \lim_{\epsilon \to 0} \epsilon/(x^2 + \epsilon^2) = \pi \delta(x) \), with \( \delta(x) \) the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases \( \int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0) \).

*note:* These concepts are important for the theory of response functions.

(4 points)