1.1.5 Equivalence relations

Consider a relation $\sim$ on a set $X$ as in ch. 1 §1.3 def. 1, but with the properties

i) $x \sim x \quad \forall x \in X$ (reflexivity)

ii) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$ (symmetry)

iii) $(x \sim y \land y \sim z) \Rightarrow x \sim z$ (transitivity)

Such a relation is called an equivalence relation. Which of the following are equivalence relations?

a) $n$ divides $m$ on $\mathbb{N}$.

b) $x \leq y$ on $\mathbb{R}$.

c) $g$ is perpendicular to $h$ on the set of straight lines $\{g, h, \ldots\}$ in the cartesian plane.

d) $a$ equals $b$ modulo $n$ on $\mathbb{Z}$, with $n \in \mathbb{N}$ fixed.

Hint: “$a$ equals $b$ modulo $n$”, or $a = b \mod(n)$, with $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$, is defined to be true if $a - b$ is divisible on $\mathbb{Z}$ by $n$; i.e., if $(a - b)/n \in \mathbb{Z}$.

(3 points)

1.1.6 Bounds for $n!$

Prove by mathematical induction that

$$n^n/3^n < n! < n^n/2^n \quad \forall n \geq 6$$

Hint: $(1 + 1/n)^n$ is a monotonically increasing function of $n$ that approaches Euler’s number $e$ for $n \to \infty$.

(4 points)

1.1.7 All ducks are the same color

Find the flaw in the “proof” of the following

Proposition: All ducks are the same color.

Proof: $n = 1$: There is only one duck, so there is only one color.

$n = m$: The set of ducks is one-to-one correspondent to $\{1, 2, \ldots, m\}$, and we assume that all $m$ ducks are the same color.

$n = m + 1$: Now we have $\{1, 2, \ldots, m, m + 1\}$. Consider the subsets $\{1, 2, \ldots, m\}$ and $\{2, \ldots, m, m + 1\}$. Each of these represent sets of $m$ ducks, which are all the same color by the induction assumption. But this means that ducks #2 through $m$ are all the same color, and ducks #1 and $m + 1$ are the same color as, e.g., duck #2, and hence all ducks are the same color.

Remark: This demonstration of the pitfalls of inductive reasoning is due to George Pólya (1888 - 1985), who used horses instead of ducks.

(2 points)

... /over
1.2.1 **Pauli group**

The Pauli matrices are complex $2 \times 2$ matrices defined as

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

Now consider the set $P_1$ that consists of the Pauli matrices and their products with the factors $-1$ and $\pm i$:

$$P_1 = \{ \pm \sigma_0, \pm i\sigma_0, \pm \sigma_1, \pm i\sigma_1, \pm \sigma_2, \pm i\sigma_2, \pm \sigma_3, \pm i\sigma_3 \}$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)
1.1.5) a) No, win it is not symmetric. E.g. 1 2 = 4, 3 1 ≠ 2.
   b) No, win it is not symmetric. E.g. 2 4 3 1 ≠ 2
   c) No, win it is not reflexive: No win is perpendicular to itself.
   d) yes.

   Proof: (i) \( a - a = 0 \) is divisible by \( n \) \( \Rightarrow a = a \mod(n) \)
   (ii) let \( a = b \mod(n) \) \( \Rightarrow \exists \lambda \in \mathbb{Z} : a - b = \lambda n \)
   \( \Rightarrow b - a = (-\lambda) n \) \( \Rightarrow b - a \) is divisible by \( n \)
   \( \Rightarrow b = a \mod(n) \)
   (iii) let \( a - b \mod(n) \) and \( b - c \mod(n) \)
   \( \Rightarrow \exists \lambda, \kappa \in \mathbb{Z} : a - b = \lambda n \) \( \land \) \( b - c = \kappa n \)
   \( \Rightarrow a - c = (a - b) + (b - c) = \lambda n + \kappa n = (\lambda + \kappa) n \)
   \( \Rightarrow a - c \mod(n) \)
   \( \Rightarrow a = b \mod(n) \) is a symmetric relation on \( \mathbb{Z} \)
1.1.6  First proof  \( n^n / 2^n < n! \) \( \neq n \geq 6 \)

\[ n = 6 : \quad 6^6 / 2^6 = 4608 < 720 = 6! \quad \checkmark \]

Assume \( n^n / 2^n < n! \)

Then

\[ \frac{(m+1)^{m+1}}{2^{m+1}} = \frac{m^m}{2^m} \left(1 + \frac{1}{m}\right)^m (m+1) \leq \frac{m^m}{2^m} \frac{e}{2} (m+1) < \frac{m^m}{2^m} (m+1) < m! (m+1) = (m+1)! \]

Now prove \( n^n / 2^n > n! \) \( \neq n \geq 6 \)

\[ n = 6 : \quad 6^6 / 2^6 = 4608 > 720 = 6! \quad \checkmark \]

Assume \( n^n / 2^n > n! \)

Then

\[ \frac{(m+1)^{m+1}}{2^{m+1}} = \frac{m^m}{2^m} \left(1 + \frac{1}{m}\right)^m (m+1) \geq \frac{m^m}{2^m} \frac{e}{2} (m+1) > m! (m+1) = (m+1)! \]

\[ \frac{n^n}{2^n} < n! < \frac{n^n}{2^n} \quad \neq n \geq 6 \]
1.1.7. The problem thus will \( n = 2 \).

The inductive step from \( n = m \) to \( n = m+1 \) relies on the fact that the subsets \( \{1, 2, \ldots, m\} \) and \( \{2, 3, \ldots, m+1\} \) have common elements. But for \( n = 2 \), we have \( m = 1 \), and hence these two sets are \( \{1\} \) and \( \{2\} \), which have no common element!

In order for the proof to be valid, we have to prove that any two subsets have the same color, which is not possible.
(iv) Each unit has a turn, viz.

\[ t_0 t_0 = t_0 \]
\[ (-t_0)(-t_0) = t_0 \]
\[ (-t_0)(-t_0) = t_0 \]
\[ (ct_0)(-ct_0) = t_0 \]
\[ (-ct_0)(ct_0) = t_0 \]

Some for \( t_1/t_2 \)