1.4.2. **The space of rank-2 tensors**

a) Prove the theorem of ch.1 §4.3: Let \( V \) be a vector space \( V \) of dimension \( n \) over \( K \). Then the space of rank-2 tensors, defined via bilinear forms \( f : V \times V \to K \), forms a vector space of dimension \( n^2 \).

b) Consider the space of bilinear forms \( f \) on \( V \) that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

*hint:* On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

1.4.3. **Cross product of 3-vectors**

Let \( x, y \in \mathbb{R}^3 \) be vectors, and let \( \epsilon_{ijk} \) be the Levi-Civita symbol. Show that the (covariant) components of the cross product \( x \times y \) are given by

\[
(x \times y)_i = \epsilon_{ijk} x^j y^k
\]

(1 point)

1.4.5. **\( \mathbb{R} \) as a metric space**

Consider the reals \( \mathbb{R} \) with \( \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( \rho(x, y) = |x - y| \). Show that this definition makes \( \mathbb{R} \) a metric space.

(3 points)

1.4.6. **Limits of sequences**

a) Show that a sequence in a metric space has at most one limit.

*hint:* Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequency with a limit is a Cauchy sequence.

(3 points)
We know that the rank-2 forms on one-to-one correspond to bilinear forms \( f(x,y) \). On the set of bilinear forms, define an addition by

\[
(f + g)(x,y) := f(x,y) + g(x,y)
\]

This makes the set of forms an additive group. Define a multiplication with scalars by

\[
(\lambda f)(x,y) := \lambda f(x,y), \quad \lambda \in k
\]

This makes the space of forms a \( k \)-vector space. On the space of rank-2 forms \( t,u \), this corresponds to defining the linear \( t+u \) as the linear vector coordinates

\[
(t+u)_{ij} = t_{ij} + u_{ij}
\]

Define the linear \( \lambda t \) as the linear vector coordinates

\[
(\lambda t)_{ij} = \lambda t_{ij}
\]

The space of linear forms is also a \( k \)-vector space.

Consider a basis \( \{e_i \} \) of \( V \), and construct \( n^2 \) twos

\[
E^i_j := e_i \otimes e_j
\]

with (under vector) coordinates

\[
(E^i_j)^{ik} = \delta^i_j \delta^k_l
\]

Define a linear form as a linear combination of the \( E^i_j \).
\[ t = \sum_{j} E_{ij} \text{ will vanish if } c \neq \lambda \]

This limit has coordinates

\[ t' = \sum_{j} E_{ij} \frac{dt}{dt} = t' \]

Any real-2 limits can be written as a linear combination of the \( E_{ij} \), will the coordinates \( t' \) of \( t \) as the weights.

\[ t = \sum_{j} E_{ij} t' \]

\( \rightarrow \) The \( E_{ij} \) span the space

In order for \( t \) to be the null vector, all of its coordinates must be zero, so \( t' = 0 \) implies \( t = 0 \) and \( E_{ij} \)

\( \rightarrow \) The \( E_{ij} \) are linearly independent

\( \rightarrow \) The \( n \times n \) matrix \( E_{ij} \) is a basis of the space of real-2 columns, i.e., the space has dimension \( n \).

b) Let \( f_{ij} \) be the bilinear form that corresponds to the basis \( E_{ij} \). Then

\[ f_{ij}(e_i, e_j) = (E_{ij})_{kl} = \delta_{ik} \delta_{kj} \text{ will be the entries } \]

For arbitrary \( x_1, y_1 \in V \) we have

\[ f_{ij}(x_1, y_1) = x_1^j y_j^1 + f(e_i, e_j) = x_1^j \delta_{ik} y_j^k = x_1^{ij} \]

\( \rightarrow \) The set of \( n^2 \) bilinear forms \( f_{ij} \) defined by

\[ f_{ij}(x_1, y_1) = x_1 y_j \]

forms a basis of the space of bilinear forms.
1.9.7. \( \text{Let } x = (x^1, x^2, x^3) \text{ and } y = (y^1, y^2, y^3) \). The cross product is defined by

\[
x \times y = (x^2 y^3 - x^3 y^2, x^3 y^1 - x^1 y^3, x^1 y^2 - x^2 y^1)
\]

On the other hand,

\[
\epsilon_{ijk} x^i y^j x^k = \left\{ \begin{array}{ll}
x^1 y^2 z^3 & i = 1 \\
x^2 y^3 z^1 & i = 2 \\
x^3 y^1 z^2 & i = 3 \\
\end{array} \right.
\]

Thus,

\[
(x \times y)^2 = \left( \sum_{i=1}^{3} x^i y^i \right)^2 = \sum_{i=1}^{3} x^i y^i \sum_{i=1}^{3} x^i y^i.
\]
Proof of the triangle inequality:

By definition of $|x|$, we have $xy = |x|1 + y \in \mathbb{R}$

$0 \leq 2(x-y)(x-y) + 2|x-y| 
\Rightarrow (x-y)^2 = x^2 - 2xy + y^2 \leq x^2 - 2xy + y^2 + 2(x-y)(x-y) + 2|x-y| 
\Rightarrow (x-y)^2 = x^2 - 2xy + y^2 + 2x^2 - 2xy - y^2 + 2 \frac{1}{2} y^2 + 2|x-y| 
\Rightarrow (x-y)^2 = x^2 - 2xy + y^2 + (x-y)^2 + 2|x-y| 
\Rightarrow (x-y)^2 = (|x-y| + |y-y|)^2 

\Rightarrow (x-y)^2 \geq 0 \Rightarrow 
\Rightarrow |x-y| \leq |x-y| + |y-y|
(4.6.10) Let $x_n$ be a sequence. Suppose $x_n \to x^*$ and $x_n \to y^*$. 

\[ f(x_n, y^*) \leq f(x^*, y^*) + f(y_n, x_n) \not\to x_n \text{ by the triangle inequality.} \]

Hence, 

\[ \lim_{n \to \infty} f(x_n, x_n) = \lim_{n \to \infty} f(y_n, x_n) = 0 \]

\[ \Rightarrow f(x^*, y^*) = 0 \Rightarrow x^* = y^* \]

b) Let $x_n$ be a limit $x^*$: $x_n \to x^*$

\[ f(x_n, x_n) \leq f(x_n, x^*) + f(x^*, x_n) \]

Let $\delta > 0$. Then $\exists N \in \mathbb{N} : f(x_n, x^*) < \delta \not\to n > N$

Now let $\varepsilon > 0$ and $\delta = \varepsilon / 2$. Then $\exists N > 0$

\[ f(x_n, x_n) \leq f(x_n, x^*) + f(x^*, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

provided $n, m > N$. \[ \Box \]
1.4.7 (c) \[ d(x-y) = \|x-y\| > 0 \quad \forall x, y \in \mathbb{R} \quad \text{by property (i) of } \|\cdot\| \]

and \[ d(x-y) = 0 \quad \text{iff } x-y = 0 \quad \Rightarrow x = y \]

\Rightarrow positive definiteness \checkmark

\[ d(y-x) = \|y-x\| = \|(x-y)\| = \|x-y\| \quad \text{by property (iii)} \]

\Rightarrow symmetry \checkmark

\[ d(x+t) = \|x+t\| = \|x-y+y+t\| \leq \|x-y\| + \|y+t\| \quad \text{by property (ii)} \]

\Rightarrow triangle inequality \checkmark

b) Consider \( \mathbb{R} \) as an \( \mathbb{R} \)-vector space and define

\[ \|x\| = |x| \quad \forall x \in \mathbb{R} \]

The \( \|\cdot\| : \mathbb{R} \to \mathbb{R} \) has all of the properties required of a norm. Furthermore, \( \|4.5x(2)\) \to 0\) as \( x \to 0\). Hence \( \mathbb{R} \) has a limit \( \Rightarrow \mathbb{R} \) is complete and hence a \( \mathbb{R} \)-space.

Same for \( \mathbb{C} \) with \( \|\cdot\| \) defined by

\[ \|z\| = |z| = \sqrt{z \bar{z}} \]

This makes \( \mathbb{C} \) a \( \mathbb{R} \)-space (among completeness)
(c) Using the definition of a norm:

(i) $$\|x\| = \sup \{ |x(t)| : \|x\| = 1 \} \Rightarrow \|x\| > 0 \text{ if } \|x\| > 0$$

The null vector is $$\mathbf{0}$$ is the null vector is defined by $$\mathbf{0}(x) = 0 \text{ for } x \in \mathbb{R}.$$

$$\Rightarrow \|\mathbf{0}\| = 0$$

Conversely, let $$\|x\| = 0.$$ Since $$\|x\| = 0$$ implies $$x = 0,$$ we find $$\mathbf{0}.$$

$$\Rightarrow ||x|| = 0 \text{ if } x = 0$$

(ii) $$\|x + y\| = \sup \{ |x(t)| + |y(t)| : \|x\| = 1 \} \leq \sup \{ |x(t)| + |y(t)| \}$$

$$\|x\| = 1$$

$$= \|x\| + \|y\|$$

Not is, $$\mathbf{x}$$ is the null vector is defined for all $$x.$$

(iii) $$\|λx\| = \sup \{ |λ(t)| : \|x\| = 1 \} = \sup \{ |λ(t)| \cdot |x(t)| \}$$

$$\|x\| = 1$$

$$= |λ| \cdot \|x\| \Rightarrow \|x\| = 0 \text{ if } x \in \mathbb{R}.$$