Problem Assignment # 5 10/27/2022
due 11/03/2022

1.4.8. **Hilbert space**

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of the definition in Problem 1.4.7. (aka the second problem on the 2021 Midterm).

   hint: Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings ℓ defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)

1.4.9. **Lorentz transformations in M_2**

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

   To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

   and determine J explicitly.

(6 points)

1.4.10. **Time-like and space-like intervals**

Consider two points (ct_x, x^1, x^2, x^3) and (ct_y, y^1, y^2, y^3) in Minkowski space. The interval between the two points is called *time-like* if

$$c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad ,$$

and *space-like* if

$$c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad .$$

Show that in interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)

. . . **/over**

1.4.11. **Special Lorentz transformations in** M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu \quad \text{(rotations)}$

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu \quad \text{(Lorentz boost along the } x\text{-direction)}$

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu \quad \text{(parity)}$

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu \quad \text{(time reversal)}$

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

1.4.8.) a) (i) $(x,x) = (x,x)^* \in \mathbb{R}$ and $(x,x) \geq 0 \rightsquigarrow \|x\| = (x,x)^{1/2} \geq 0 \ \forall x \in H$

1/2

and $(x,x) = 0$ iff $x = \emptyset \rightsquigarrow \|x\| = 0$ iff $x = \emptyset$

(ii) $\|x+y\|^2 = (x+y, x+y) = (x,x) + (y,y) + (x,y) + (y,x)$

$= (x,x) + (y,y) + (x,y) + (x,y)^*$

$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,(x,y)$

But $(\mathrm{Re}\,(x,y))^2 + (\mathrm{Im}\,(x,y))^2 = |(x,y)|^2$.

$\rightsquigarrow \mathrm{Re}\,(x,y)^2 = |(x,y)|^2 - \underbrace{(\mathrm{Im}\,(x,y))^2}_{\geq 0} \leq |(x,y)|^2$

CS (§4.7 lemma)

$\leq (x,x)(y,y) = \|x\|^2 \cdot \|y\|^2$

$\rightsquigarrow \|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2$

1

$\rightsquigarrow \|x+y\| \leq \|x\| + \|y\|$

(iii) $\|ax\|^2 = (ax, ax) = aa^*(x,x) = |a|^2 \cdot \|x\|^2$

1/2

$\rightsquigarrow \|ax\| = |a| \cdot \|x\| \quad \forall a \in \mathbb{C}, x \in H \quad \rightsquigarrow \|\ldots\|$ is a norm

b) The definition $l(x) := (y,x)$ implies

(i) $l(x+z) = (y, x+z) = (y,x) + (y,z) = l(x) + l(z) \quad \forall x,z \in H$

(ii) $l(\lambda x) = (y, \lambda x) = \lambda (y,x) = \lambda\, l(x) \quad \forall x \in H, \lambda \in \mathbb{C}$

1

$\rightsquigarrow l$ is a linear form

1.4.9, a) Let $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For Λ to be a Lorentz transfo, we must have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2-b^2 & ac-bd \\ ac-bd & c^2-d^2 \end{pmatrix}$$

From this obtain three constraints on the four numbers a, b, c, d

(i) $a^2-b^2=1$

(ii) $c^2-d^2=-1$

(1) (iii) $ac-bd=0$

Now consider $\sinh\phi$, which maps $\mathbb{R}$ one-to-one onto itself

$\rightsquigarrow \forall\, b\in\mathbb{R}\ \exists!\, \phi\in\mathbb{R}:\quad \underline{b=\sinh\phi}$

(i) $\rightsquigarrow\ a^2 = 1+b^2 = 1+\sinh^2\phi = \cosh^2\phi \ \rightsquigarrow\ \underline{a=\sigma\cosh\phi},\ \underline{\sigma=\pm1}$

Analogously, $\underline{c=\sinh\psi}$

(1) (ii) $\rightsquigarrow\ d^2 = 1+c^2 = 1+\sinh^2\psi = \cosh^2\psi \ \rightsquigarrow\ \underline{d=\tau\cosh\psi},\ \underline{\tau=\pm1}$

Finally,

(iii) $\rightsquigarrow\ 0 = \sigma\cosh\phi\sinh\psi - \tau\cosh\psi\sinh\phi$

$= \cosh\phi\sinh\psi - \sigma\tau\cosh\psi\sinh\phi$

$= \cosh(\sigma\tau\phi)\sinh\psi - \cosh\psi\sinh(\sigma\tau\phi)$

(1) $= \sinh(\psi-\sigma\tau\phi) \quad\rightsquigarrow\quad \underline{\psi=\sigma\tau\phi}$

$$\rightsquigarrow\ \underline{\underline{\Lambda_{\sigma,\tau}(\phi)}} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sinh(\sigma\tau\phi) & \tau\cosh(\sigma\tau\phi) \end{pmatrix} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \tau\sigma\sinh\phi & \tau\cosh\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}\Lambda(\phi)\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}}} \quad \text{with} \quad \underline{\underline{\Lambda(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix},\ \phi\in\mathbb{R}}}$$

(1) and $\sigma,\tau=\pm1$ is the most general element of $O(1,1)$.

b) (i) $\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix}\begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$

$\rightsquigarrow D(\phi_1)D(\phi_2) = D(\phi_1+\phi_2)$ closure ✓

(ii) Matrix multiplication is associative ✓

(iii) $D(\phi=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$ neutral element ✓

(iv) $D(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$

and $\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$\rightsquigarrow D(-\phi) = (D(\phi))^{-1}$ inverse ✓

$\rightsquigarrow$ $\{D(\phi)\}$ is a group $SO^+(1,1)$

1 (i), (iii), (iv) provide the isomorphism $SO^+(1,1) \cong R(+)$

c) $e^{J\phi} = \mathbb{1}_2 + J\phi + \frac{1}{2}J^2\phi^2 + \ldots$

and $\cosh\phi = 1 + \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 + \ldots$, $\sinh\phi = \phi + \frac{1}{3!}\phi^3 + \ldots$

Try $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\rightsquigarrow$ $J^2 = \mathbb{1}_2$, $J^3 = J$, etc.

$\rightsquigarrow e^{J\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\phi^2 & 0 \\ 0 & \frac{1}{2}\phi^2 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{3!}\phi^3 \\ \frac{1}{3!}\phi^3 & 0 \end{pmatrix} + \ldots$

$= \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$ ✓

1 $\rightsquigarrow$ J is the generator of $SO^+(1,1)$.

1.4.10.) Let $$c^2(t_x-t_y)^2 \geq (x^1-y^1)^2+(x^2-y^2)^2+(x^3-y^3)^2$$

$$\Leftrightarrow \quad \boxed{(x^\mu-y^\mu)\,g_{\mu\nu}\,(x^\nu-y^\nu) \geq 0} \qquad (*)$$

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Now consider a Lorentz trafo $x^\mu = \Lambda^\mu{}_\nu\, \tilde{x}^\nu$

$$(*) \rightsquigarrow \quad (\tilde{x}^\alpha-\tilde{y}^\alpha)\,\underbrace{\Lambda^\mu{}_\alpha\, g_{\mu\nu}\, \Lambda^\nu{}_\beta}_{=g_{\alpha\beta}}\,(\tilde{x}^\beta-\tilde{y}^\beta) \geq 0$$

$$\rightsquigarrow \quad (\tilde{x}^\mu-\tilde{y}^\mu)\,g_{\mu\nu}\,(\tilde{x}^\nu-\tilde{y}^\nu) \geq 0$$

$\rightsquigarrow$ (*) holds in all inertial frames

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1.9.11 a) i) $\underline{R^\mu{}_\alpha\, g_{\mu\nu}\, R^\nu{}_\beta} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^T \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^T \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -R \end{array}\right)$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -R^TR \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \underline{\underline{g_{\alpha\beta}}} \quad ✓$$

ii) $\underline{\Lambda^\mu{}_\alpha\, g_{\mu\nu}\, \Lambda^\nu{}_\beta} = \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\times \begin{pmatrix} \cosh\alpha & 0 & 0 & \sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cosh^2\alpha - \sinh^2\alpha & 0 & 0 & \cosh\alpha\sinh\alpha - \sinh\alpha\cosh\alpha \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sinh\alpha\cosh\alpha - \cosh\alpha\sinh\alpha & 0 & 0 & \sinh^2\alpha - \cosh^2\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \underline{\underline{g_{\alpha\beta}}} \quad ✓$$

iii) $\underline{P^\mu{}_\alpha\, g_{\mu\nu}\, P^\nu{}_\beta} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right) = \underline{\underline{g_{\alpha\beta}}} \quad ✓$

iv) $\underline{T^\mu{}_\alpha\, g_{\mu\nu}\, T^\nu{}_\beta} = (-)^2 P^\mu{}_\alpha\, g_{\mu\nu}\, P^\nu{}_\beta = \underline{\underline{g_{\alpha\beta}}} \quad ✓$

b) $R^\mu{}_\nu$ leaves the time coordinate invariant, and we know that the $R^i{}_j$ form a group (see PH 6.1.1) $\rightarrow \{R^i{}_j\}$ is a subgroup of L.
The Lorentz boosts $\Lambda^\mu{}_\nu$ are a subgroup of L according to Problem 1.8.

c) group tables:

	1	P
1	1	P
P	P	1

and

	1	T
1	1	T
T	T	1

$\rightarrow \{1, P\}$, and $\{1, T\}$ are subgroups of L under matrix multiplication

	1	P	T	PT
1	1	P	T	PT
P	P	1	PT	T
T	T	PT	1	P
PT	PT	T	P	1

$\rightarrow \{1, P, T, PT\}$ is also a subgroup of L