2.2.3. **Electrostatics in $d$ dimensions (continued)**

This is a continuation of Problem #2.2.3.

b) Calculate and plot the potential $\phi$ and the field $E$ for $d = 2$ for the case of a homogeneously charged disk, $\rho(x) = \rho_0 \Theta(r_0 - |x|)$.

*hint:* It is easiest to proceed as in the 3-$d$ case, see Problem 2.2.2.

*note:* This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0 \Theta(x_0^2/4 - x^2)$.

*hint:* Integrate Poisson’s formula directly.  

(8 points)

2.2.4. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \nabla^2) \varphi(x) = 4\pi \rho(x)$$

in terms of an integral.

*hint:* The answer is a generalization of Poisson’s formula.  

(3 points)

2.3.1. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$.

Show that the potential to that order is given by

$$\varphi(x) = \frac{1}{r} Q + \frac{1}{r^3} x \cdot d + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \ldots$$

with $Q$ the total charge and $d$ the dipole moment, and determine the quadrupole tensor $Q_{ij}$.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor $Q_{ij}$ with respect to the ellipsoid’s center. Check to make sure that the result for $Q_{ij}$ is traceless.

d) Let the charge density be invariant under rotations about the $z$-axis through multiples of an angle $\alpha$, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form

$$\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}.$$ 

Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from part c) and calculate the quadrupole moments $Q_{2m}$ as defined in ch.2 §3.5.

(10 points)