2.3.5. **Field due to distant charges**

Consider the electric field generated by a charge density $\rho(y)$ that vanishes inside a sphere with radius $r_0$: $\rho(y) = 0$ for $|y| \leq r_0$. Show that

(a) If $\rho$ is invariant under parity operations, $\rho(-y) = \rho(y)$, then the electric field at the origin vanishes.

(b) If $\rho(y)$ is invariant under rotations about the $z$-axis through multiples of an angle $\alpha$ with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form

$$\phi_{ij}(x = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$$

(c) If $\rho(y)$ has cubic symmetry, i.e., if $\rho(y)$ is invariant under rotations through $\pi/2$ about any of the three axes $x$, $y$, and $z$, then the field-gradient tensor at the origin vanishes.

(6 points)

2.3.7. **Electrostatic interaction: Quadrupole in an external electric field**

Consider the following classical model for a nuclear quadrupole moment in a crystal lattice: A rectangular parallelepiped (height $A$, length and width $B$) carries a charge $e$ at each of its eight corners. At the center of the parallelepiped is a homogeneously charged spheroid (charge $Q$, semi-axes $a$ and $b$). The symmetry axis of the spheroid forms an angle $\theta$ with the $A$-axis of the parallelepiped. The center of the spheroid is fixed, but the angle $\theta$ can vary. Let $A \gg a$, $B \gg b$.

(a) Calculate the electrostatic interaction energy $U$ of this system to quadrupolar order. Show that $U$ can be expressed in terms of $e$, the lattice constants $A$ and $B$, and the quadrupole moment $Q_{33}$ of the spheroid in the coordinate system of the lattice.

(b) Calculate the quadrupole moment $Q'_{33}$ of the spheroid in its principal-axes system, and then calculate $Q_{33}$ by transforming into the lattice system. Express $U$ as a function of the angle $\theta$.

*hint:* In general, lining up the principal-axes systems would require three Euler angles. However, due to the symmetries of the problem $Q'_{33}$ and $Q_{33}$ in the present case are related by only one angle, viz., $\theta$.

(c) Find the equilibrium positions of the spheroid. Make sure to distinguish the cases of prolate and oblate spheroids ($a > b$ and $a < b$, respectively), as well as between the cases $A > B$ and $A < B$.

(15 points)
\[ \psi(x) = \int \frac{d^3k}{|x-k|^4} \quad \frac{\gamma(k)}{|x-k|^2} = \psi(x=0) + \psi \nabla \psi \mid_{x=0} + \frac{1}{2} \nabla \cdot \nabla \psi \mid_{x=0} + \ldots \]

\[ = \psi_0 - \nabla \cdot \mathbf{E} + \frac{1}{2} \nabla \cdot \nabla \psi \mid_{x=0} + \ldots \]

\[ = \psi_0 + \psi_k(x) + \psi_L(x) + \ldots \]

\[ \text{a) } \gamma(k) \sim \gamma(-k) \rightarrow \psi(-x) = \int d^3k \frac{\gamma(k)}{|x-k|^2} = \int d^3k \frac{\gamma(-k)}{|x-k|^2} = \psi(x) \]

\[ \rightarrow \text{All } m \text{ odd } \psi \text{ vanish, & pochier } \mathbf{E} = 0 \]

\[ \text{b) } \psi_{ij} \text{ is not symmetric } \rightarrow \text{ ADerived RE} \text{ale & let } \psi_{ij} \text{ is diagonal} \]

\[ \phi(x) \text{ obeys Laplace's eq, & } |x| < r_0 \]

\[ \rightarrow \nabla \phi = 0 \]

\[ \rightarrow \phi_{ij} \text{ has the form } \phi_{ij} = \begin{pmatrix} \phi_+ + \phi_- & 0 & 0 \\ 0 & \phi_+ - \phi_- & 0 \\ 0 & 0 & -2\phi_+ \end{pmatrix} \]

\[ \text{where } \phi_0 = \frac{1}{2} (\phi_{xx} - \phi_{zz}) \]

\[ \rightarrow \psi(x) = \frac{1}{2} x^2 \nabla^2 \psi (\phi_+ + \phi_-) + \frac{1}{2} x^2 \nabla \cdot \nabla \psi (\phi_+ - \phi_-) + \frac{1}{2} x^2 \nabla \cdot \nabla \psi (-2\phi_+) \]

\[ = \frac{1}{2} x^2 \left[ (1 - 2\mu^2) \phi_+ + \mu^2 x^2 \left( \phi_+ + \phi_- \right) \right] \]

Rational invariance of \( \phi(x) \) implies rational invariance of \( \psi(x) \), & \( \nabla \phi \) pochier of \( \psi_L(x) \)
\[ \psi_2(r, \varphi, \psi + \delta \varphi) = \frac{1}{2} r^2 \left[ 1 - 2m^2 \right] \psi_+ + \frac{1}{2} r^2 \left[ 1 - 2m^2 \right] \psi_- \]

\[ \psi_2(r, \varphi, \psi) = \psi_2(r, \varphi, \psi) \]

\[ \psi_+ \left( 2 \psi + 2 \delta \varphi \right) = \psi_+ \left( 2 \psi + 2 \varphi \right) \rightarrow \psi_+ = 0 \]

(1) Let \( \mathcal{L} \) be a \( \mathbb{Z}_2 \)-invariant action rotating \( \varphi \) about any of the three axes \( x, y, z \).

(2) \( \psi_+ = 0 \) due to \( \mathcal{L} \)-invariance of \( \psi_+ \).

(3) \( \mathcal{L} \)-invariance of \( \mathcal{L} \) implies \( \mathcal{L} \)-invariance of \( \varphi(x) \).

\[ \psi_2(r, \varphi + \delta \varphi, \psi) = \frac{1}{2} r^2 \psi_+ \left[ 1 - 2m^2 \right] \varphi + \frac{1}{2} r^2 \psi_- \left[ 1 - 2m^2 \right] \varphi \]

\[ \psi_+ \left( 2 \psi + 2 \delta \varphi \right) = \psi_+ \left( 2 \psi + 2 \varphi \right) \rightarrow \psi_+ = 0 \rightarrow \psi_+ = 0 \]
2.27. (a) Consider the problem due to two charges:

\[ \varphi(x) = e \frac{\xi}{x_1^{(21)}} \left( \frac{1}{x_1^{(21)}} \right) \quad \text{when} \quad x_1^{(21)} = \frac{1}{2} \left( \frac{x_1}{x_1} \right) \]

We have

\[ \varphi_0 = \varphi(x=0) = e \frac{\xi}{x_1^{(21)}} \left( \frac{1}{x_1^{(21)}} \right) = e \frac{\xi}{1^{2} + 2 \xi^{2}} = \frac{16e}{1^{2} + 2 \xi^{2}} \]

\[ E = -\nabla \varphi(x=0) = \frac{\xi}{x_1^{(21)}} - \frac{2(21)}{x_1^{(21)}} = 0 \]

\[ \varphi_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \varphi \bigg|_{x=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\varphi \\ 0 & -\varphi & 0 \end{pmatrix} \]

b) symmetry, see Problem 2.3.5 b)

\[ \psi = \psi_{ij} = \frac{\partial^2}{\partial x_i} \left( e \frac{\xi}{x_1^{(21)}} \right) \left( \frac{1}{x_1^{(21)}} \right) = e \left( \frac{\xi^{(21)}}{x_1^{(21)}} \right) \left( \frac{1}{x_1^{(21)}} \right) \]

Minin \( r_0 := \sqrt{4^2 + \xi^2} \rightarrow l^{(21)} = \xi r_0 \)

\[ \Rightarrow \varphi_0 = \frac{16e}{r_0} \]

\[ \psi = e \left( \frac{\xi^{(21)}}{r_0} \right) \left( \frac{1}{r_0} \right) \]

\[ = e \frac{1}{r_0^2} \left( \frac{2 \xi^{(21)} + 8 \xi_1 - 2 \xi_0^2}{2 \xi^{(21)} + 8 \xi_1 - 2 \xi_0^2} \right) = \frac{2e}{r_0^2} \left( \frac{2 \xi^{(21)} + 8 \xi_1 - 2 \xi_0^2}{2 \xi^{(21)} + 8 \xi_1 - 2 \xi_0^2} \right) = \frac{2e}{r_0^2} \left( \xi^{(21)} \right) \]

\[ \Rightarrow \mu = \varphi_0 \xi + \frac{1}{2} \left( \psi_a \psi_a + \psi_a \psi_a - 2 \psi \psi_a \right) \]

\[ = \varphi_0 \xi - \psi \psi_a \]

\[ \sum_{a} \psi_{aa} = 0 \quad \Rightarrow \varphi_0 \xi - \psi \psi_a = 0 \]

\[ \sum_{a} \psi_{aa} = \varphi_0 \xi \]
mark: the \( Q_{ij} \) is the quadrupole moment of the spherical
lattice coordinate system!

b) in the principal-axis system of the spherical quadrupole,
the tensor has the form

\[
Q'_{ij} = \begin{pmatrix}
\frac{9}{4} & 0 & 0 \\
0 & \frac{9}{4} & 0 \\
0 & 0 & -2g
\end{pmatrix}
\]

Transform to the lattice system by means of rotation
matrices (angles of 10/21) \( \phi \)

\[
Q_{ij} = \Sigma Q'_{ij} D_{\phi} D_{\psi} D_{\chi}
\]

\[
Q_{22} = Q'_{22} D_{\psi}^2 + Q'_{22} D_{\phi} D_{\chi} + Q'_{22} D_{\phi}^2 D_{\chi}^2
\]

\[
= g \left( D_{\psi}^2 \right) + \left( D_{\phi} D_{\chi} \right)^2 - 2g \left( D_{\phi}^2 \right) \\
= \frac{1}{4} \left[ \left( D_{\psi}^2 \right) \left( D_{\phi}^2 \right) \left( D_{\chi}^2 \right) \right]
\]

Now \( Q_{ij} \) is an orthogonal tensor \( \Rightarrow D_{\psi}^2 + D_{\phi}^2 + D_{\chi}^2 = 1 \)

and \( \phi \) must align the \( 2' \)-axis with the \( \phi \)-axis \( \rightarrow D_{\phi} = 0 \)

\[
Q_{22} = g \left[ 1 - D_{22}^2 - 2D_{22}' \right] = g \left[ 1 - 2u_2 \right]
\]

Finally, Problem 2.3.1 with \( q = \frac{Q}{10} \left( 6^2 - 5^2 \right) \)

\[
U = Q \alpha - \frac{2e}{\alpha} \left( \begin{pmatrix} 6^2 \\ -5^2 \end{pmatrix} \right) g \left( 1 - 2u_2 \right)
\]

\[
= Q \alpha + \frac{2e}{\alpha} \left( \begin{pmatrix} 6^2 \\ -5^2 \end{pmatrix} \right) \left( 6^2 - 5^2 \right) \left( 2u_2 \right)
\]
9) \( \frac{1}{x-1} \) is minimum for \( x = 0 \)

\[ \Rightarrow \quad x = \frac{1}{2} \]

minimum for \( x = \pm \frac{1}{2} \)

\[ \Rightarrow \quad x = 0, \frac{1}{2} \]

If \( c > 0 \) \( \Rightarrow \) \( u \) is minimum for

\[ x = \frac{1}{2} \quad \text{if} \quad (a^2 - c^2)(c^2 - b^2) > 0 \]

\[ x = 0 \quad \text{if} \quad (a^2 - c^2)(c^2 - b^2) < 0 \]

prolate spheroid \((a > b)\)

( major )

oblate \((a < b)\): flips the two cones

(disc)

\( c < 0 \): flips the two cones each