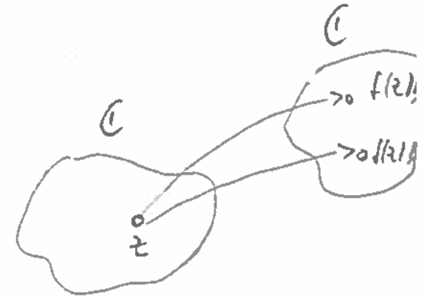
§2 Complex-valued functions of complex arguments

Consider the field $\mathbb{C}$ of complex numbers $z = z_1 + i z_2 \equiv z' + i z''$ ($z_1, z_2, z', z'' \in \mathbb{R}$) as constructed in ch I §3.]

2.1 Complex functions

def. 1: (a) Let $f: \mathbb{C} \to \mathbb{C}$ be a mapping in the sense of ch I §1.2. Then we call f a (single-valued) complex-valued function of a complex argument.



(b) Generalize the concept of a mapping such that one pre-image can have $n \in \mathbb{N}$ images. Then we call f a (n-valued) function.

example: (1) $f(z) = z^2$ is a single-valued function.

(2) $f(z) = z^{1/2}$ is a two-valued function.

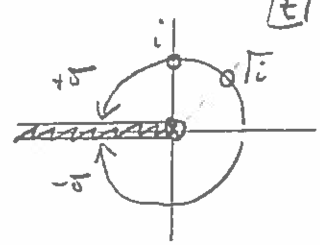
(3) $f(z) = e^z := \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ is a single-valued function.

(4) $z = re^{i\varphi} \equiv re^{i(\varphi + 2\pi n)}$, $n \in \mathbb{Z}$

$$\log z = \log\left(re^{i(\varphi+2\pi n)}\right) = \log r + \log e^{i(\varphi+2\pi n)} = \log r + i(\varphi + 2\pi n)$$

is a $\mathbb{Z}$-valued function.

def. 2: A multivalued function $f(z)$ can be made single-valued by cutting the complex plane along a branch cut chosen such that the function remains single-valued if the cut is not crossed.



example: (5) Make $f(z)=z^{1/2}$ single-valued by choosing the cut along the negative real axis. Then $i^{1/2}=e^{i\pi/4}$ uniquely, etc.

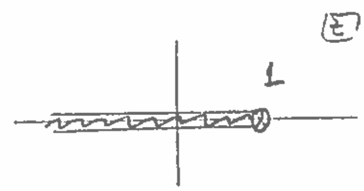
(6) $f(z)=\log z$ can be made single-valued by choosing the same branch cut.

remark: (1) The branch cut is a property of the individual function, not of the complex plane in general.

(2) For a given function, the choice of the branch cut is not unique. For instance, choosing the cut in examples (5),(6) along the positive real axis corresponds to $\varphi \in [0, 2\pi[$.

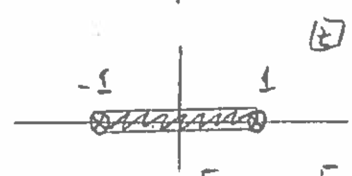
(3) For functions of the form $f(g(z))$ the branch cut will start at a "branch point" z_0 determined by $g(z_0)=0$, rather than at the origin.

example: (7) $f(z)=\log(z-1)$



(8) $f(z)=\log\frac{z-1}{z+1}$

$= \log(z-1)-\log(z+1)$



The branch cuts cancel each other for $[-\infty,-1[$ (see Problem ~~ϕ~~ 22).

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$d=3,2$

def. 3: For a two-valued function, we can continue the function across the cut onto a second sheet, on which the function takes on the other possible value. The two sheets will cover the entire complex plane 2-fold and form the Riemann surface for the function. An analogous construction works for an n-valued function.

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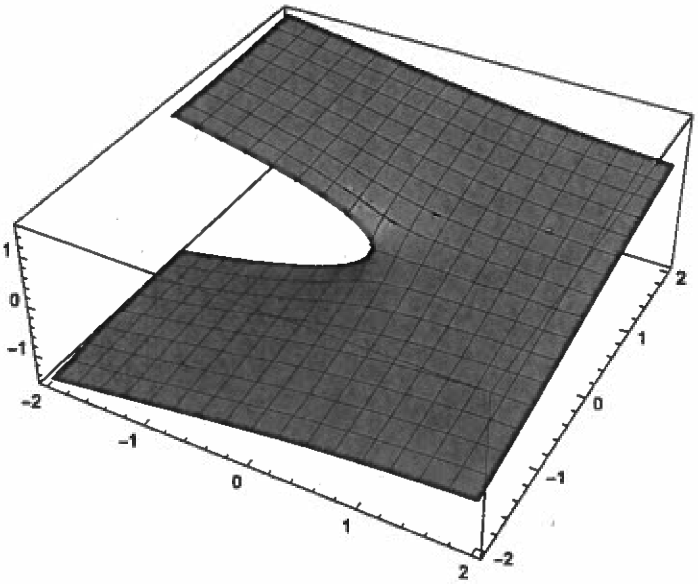
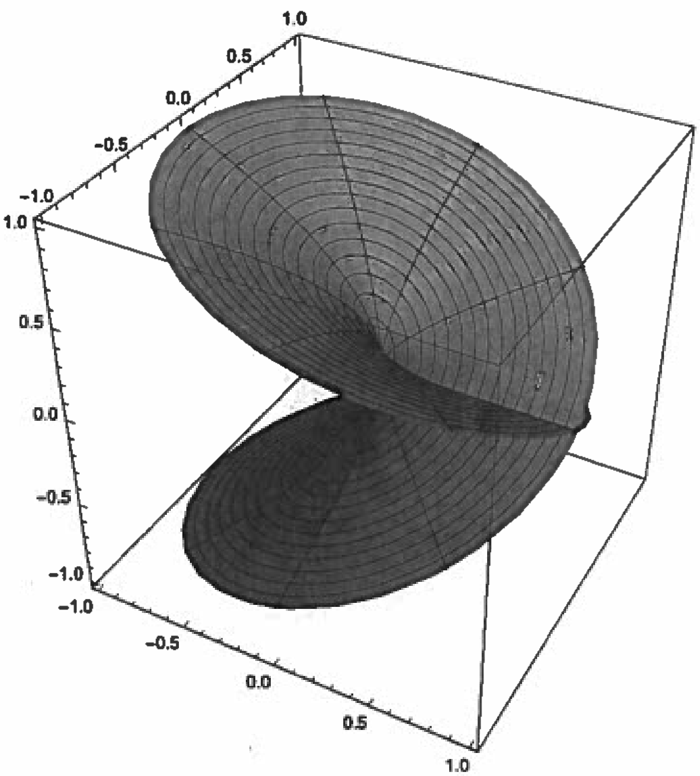
In[11]:= `Plot3D[Im[Sqrt[x + I y]], {x, -2, 2}, {y, -2, 2}, PlotPoints → 30]`

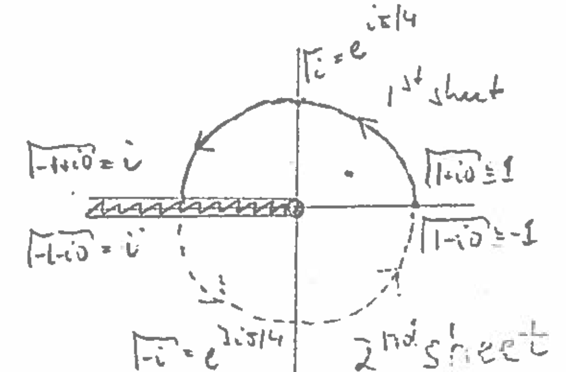
Out[11]=

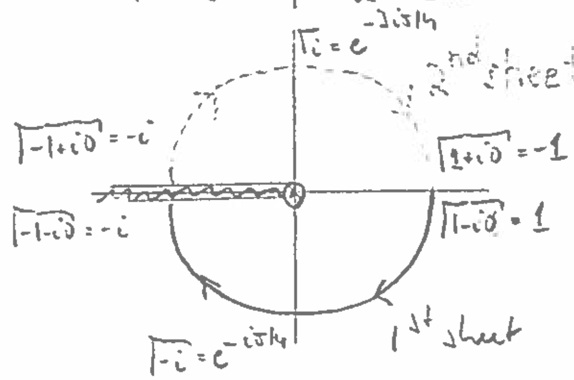


In[14]:= `ParametricPlot3D[{r Cos[phi], r Sin[phi], Sqrt[r] Sin[phi/2]},`
`{r, 0, 1}, {phi, 0, 4 Pi}, PlotPoints → {20, 60}]`

Out[14]=

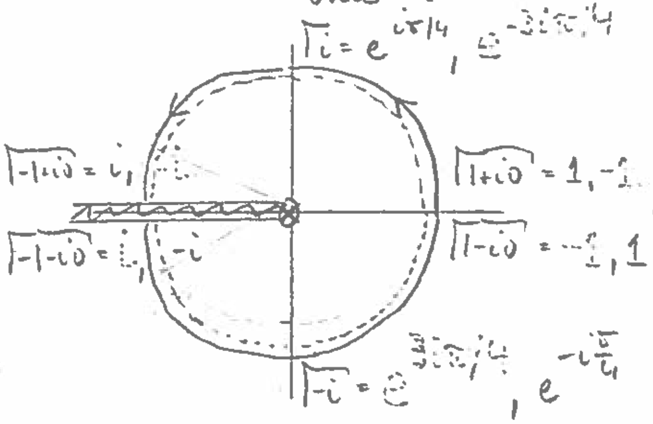






example: (a) $f(z) = z^{1/2}$

continuation on either sheet past the cut brings one back to the other sheet:



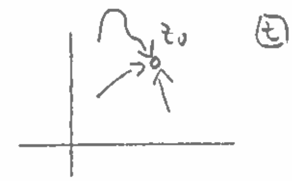
2.2 Analyticity

def 1: $f(z)$ is called continuous in the point $z_0 \in \mathbb{C}$ if $f(z_0)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$.

def. 2: $f(z)$ is called differentiable in z_0 with derivative $df/dz|_{z_0}$ if the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: \frac{df}{dz}\Big|_{z_0}$ exists.

remark: (1) These are obvious generalizations of the corresponding concept for real functions.

(2) The limit must exist no matter how z approaches z_0 in the complex plane!



→ These are much stronger requirements than the corresponding ones for real-valued functions of one real argument.

def. 3: Let $R \subset \mathbb{C}$ be a region in $\mathbb{C}$ and $f: R \to \mathbb{C}$ a function. f is called analytic on R if it is differentiable in all points $z \in R$.

Theorem: Cauchy - Riemann

$f(z) = f'(z',z'') + i f''(z',z'')$ is analytic in R iff

$$(+) \quad \boxed{\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \quad \text{and} \quad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}} \quad \forall z \in R$$

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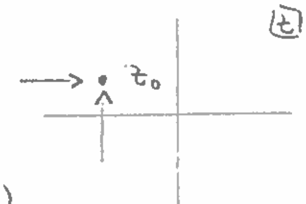
proof: Consider the difference quotient $\frac{\Delta f}{\Delta z} = \frac{f'(z',z'') + i f''(z',z'') - f'(z_0',z_0'') - i f''(z_0',z_0'')}{z' + i z'' - z_0' - i z_0''}$.

For $\frac{df}{dz} = \lim_{z \to z_0} \frac{\Delta f}{\Delta z}$ to exist, the limit must exist for f' and f'' separately, and it must exist for $z \to z_0$ along either the real or the imaginary axis $\Rightarrow$ If $f(z)$ is differentiable in z_0, then the partial derivatives $\partial f'/\partial z'$, $\partial f'/\partial z''$, $\partial f''/\partial z'$, $\partial f''/\partial z''$ exist.



Now approach z_0 along the direction parallel to the real axis, i.e., for fixed z''. $\Rightarrow$

$$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z''=\text{const}}} = \frac{\partial f'}{\partial z'} + i \frac{\partial f''}{\partial z''} \quad (*)$$

skip

and now for fixed z':

$$\left.\frac{df}{dz}\right|_{\substack{z_0 \\ z'=\text{const}}} = \frac{1}{i}\frac{\partial f'}{\partial z''} + \frac{i}{i}\frac{\partial f''}{\partial z''} = \frac{\partial f''}{\partial z''} - i \frac{\partial f'}{\partial z''} \quad (**)$$

But if f is differentiable, then $(*) = (**)$, and hence

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \quad \text{and} \quad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}$$

Now we need to prove the converse. We have

$$f(z) - f(z_0) = f'(z',z'') + i f''(z',z'') - f'(z_0',z_0'') - i f''(z_0',z_0'')$$

But from Taylor's theorem we have, for $z \to z_0$,

$$f'(z',z'') - f'(z_0',z_0'') \to \left.\frac{\partial f'}{\partial z'}\right|_{z_0}(z'-z_0') + \left.\frac{\partial f'}{\partial z''}\right|_{z_0}(z''-z_0'')$$

and $$f''(z',z'') - f''(z_0',z_0'') \to \left.\frac{\partial f''}{\partial z'}\right|_{z_0}(z'-z_0') + \left.\frac{\partial f''}{\partial z''}\right|_{z_0}(z''-z_0'')$$

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$$\rightsquigarrow \frac{f(z)-f(z_0)}{z-z_0} \to \frac{1}{z-z_0}\Big[\frac{\partial f'}{\partial z'}\Big|_{z_0}(z'-z_0') + \frac{\partial f'}{\partial z''}\Big|_{z_0}(z''-z_0'') + i\frac{\partial f''}{\partial z'}\Big|_{z_0}(z'-z_0') + i\frac{\partial f''}{\partial z''}\Big|_{z_0}(z''-z_0'')$$

$$\overset{(*)}{=} \frac{1}{z-z_0}\Big[\frac{\partial f'}{\partial z'}\Big|_{z_0}(z'-z_0'+iz''-iz_0'') + \frac{\partial f'}{\partial z''}\Big|_{z_0}(z''-z_0''-iz'+iz_0')\Big]$$

$$= \frac{\partial f'}{\partial z'}\Big|_{z_0}\frac{z-z_0}{z-z_0} - i\frac{\partial f'}{\partial z''}\frac{z-z_0}{z-z_0} \overset{(*)}{=} \underline{\frac{\partial f'}{\partial z'}\Big|_{z_0} + i\frac{\partial f''}{\partial z'}\Big|_{z_0}}$$

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The rhs exists, and therefore $\frac{df}{dz}\big|_{z_0}$ exists and is independent of how the limit is taken. □

example: (1) $f(z) = z^2 = \underbrace{(z'^2 - z''^2)}_{=f'} + i\underbrace{2z'z''}_{=f''}$ $\qquad \frac{\partial f'}{\partial z'} = 2z' = \frac{\partial f''}{\partial z''}$, $\quad \frac{\partial f'}{\partial z''} = -2z'' = -\frac{\partial f''}{\partial z'}$

$\rightsquigarrow$ f is analytic on $\mathbb{C}$.

skip the algebra

(2) $f(z) = \frac{1}{z} = \frac{z'}{z'^2+z''^2} - i\frac{z''}{z'^2+z''^2} \rightsquigarrow f' = \frac{z'}{|z|^2},\ f'' = \frac{-z''}{|z|^2}$

$$\frac{\partial f'}{\partial z'} = \frac{|z|^2-2z'^2}{|z|^4}, \quad \frac{\partial f''}{\partial z''} = -\frac{|z|^2-2z''^2}{|z|^4} = \frac{-z'^2+z''^2}{|z|^4} = \frac{|z|^2-2z'^2}{|z|^4}$$

$$\frac{\partial f'}{\partial z''} = \frac{-2z'z''}{|z|^4}, \quad \frac{\partial f''}{\partial z'} = -\frac{-2z'z''}{|z|^4} = \frac{2z'z''}{|z|^4}$$

$\rightsquigarrow$ f is analytic on $\mathbb{C}\setminus\{0\}$

(3) $f(z) = 1/(z-z_0)^n$, $n\in\mathbb{N}$ is analytic on $\mathbb{C}\setminus\{z_0\}$.

corollary: If $f: R \to \mathbb{C}$ is analytic, then f' and f'' satisfy Laplace's differential equation

$$\frac{\partial^2\varphi}{\partial z'^2} + \frac{\partial^2\varphi}{\partial z''^2} = 0 \qquad \varphi = f', f''$$

everywhere in R.

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proof: Cauchy-Riemann $\rightsquigarrow$ $\underline{\frac{\partial^2 f'}{\partial z'^2}} = \frac{\partial^2 f''}{\partial z''\partial z'} = \underline{-\frac{\partial^2 f'}{\partial z''^2}}$

and $\underline{\frac{\partial^2 f''}{\partial z'^2}} = -\frac{\partial^2 f'}{\partial z''\partial z'} = \underline{-\frac{\partial^2 f''}{\partial z''^2}}$ □

remark: (1) Here we have assumed that the second derivatives exist. One can show that they do, and so do all higher derivatives! This is another indication of how