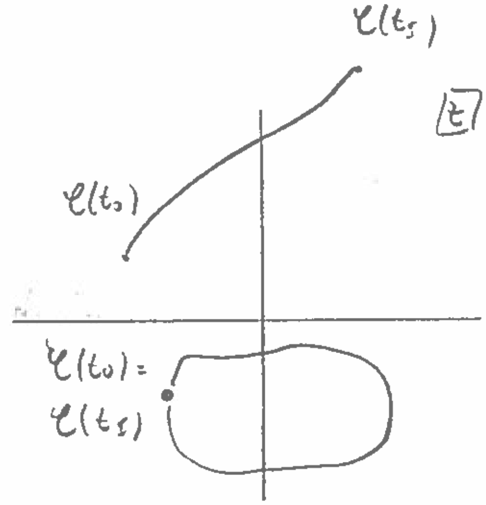
§3 Integration in the complex plane

3.1 Path integrals



Let $J := [t_0, t_1] \subset \mathbb{R}$ be a real interval and define paths in $\mathbb{C}$ with parameter t as in §1.2 by means of the isomorphism $\mathbb{R}^2 \cong \mathbb{C}$. If $\mathcal{C} = \{z(t), t \in J\}$ we will write $\mathcal{C}(t)$ instead of $z(t)$. If $\mathcal{C}(t_0) = \mathcal{C}(t_1)$ we call the path closed.

def. 1: Let $R \subset \mathbb{C}$ and let $\mathcal{C}$ be a path such that $\mathcal{C}(t) \in R \;\forall t \in J$. Let $f : R \to \mathbb{C}$ be a complex function. Then we define the path integral of f along $\mathcal{C}$ by

$$\boxed{\int_{\mathcal{C}} dz\, f(z) := \int_{t_0}^{t_1} dt\, f(\mathcal{C}(t))\, \frac{d\mathcal{C}}{dt}}$$

with the r.h.s. an ordinary Riemann integral over J.

remark: (0) Note that this is similar to, but different from, §1.2 def. 3.

def. 2: Let $f : R \to \mathbb{C}$ be a complex function. $F : R \to \mathbb{C}$ is called an indefinite integral of f if F is analytic on R and

$$\left.\frac{dF}{dz}\right|_{z_0} = f(z_0) \quad \forall z_0 \in R.$$

def. 3: A region $R \subset \mathbb{C}$ is called simply connected if any path in R can be continuously deformed to a point.

remark: (1) "simply connected" means "no holes"

 is simply connected.  is not

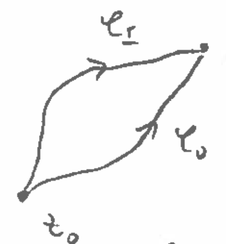
theorem: Cauchy's integral theorem

Let $R \subset \mathbb{C}$ be simply connected, and let $f: R \to \mathbb{C}$ be analytic on R. Then f has an indefinite integral and

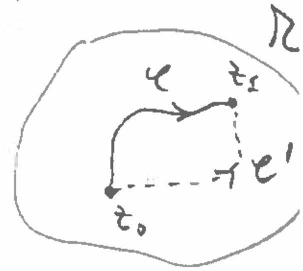
$$\boxed{\int_{\mathcal{C}} dz\, f(z) = F(z_1) - F(z_0)}$$

for all paths $\mathcal{C}: [t_0, t_1] \to \mathbb{C}$ with $\mathcal{C}(t_0) = z_0$ and $\mathcal{C}(t_1) = z_1$.

remark: (1) This says in particular that the integral depends only on the starting and end points of $\mathcal{C}$. For fixed z_0 and z_1 it does not depend on $\mathcal{C}$.



(2) This implies that we can deform the integration contour, keeping the starting and end points fixed, without changing the value of the integral, as long as we stay within a region where f is analytic.



proof: see, e.g., Smirnov III/2 Sec. 4, or Whittaker & Watson ch. (not difficult, but lengthy).

corollary: Let $R \subset \mathbb{C}$ be simply connected, and $f: R \to \mathbb{C}$ analytic and $\mathcal{C}$ a closed path in R. Then

$$\boxed{\oint_{\mathcal{C}} dz\, f(z) = 0}$$

proposition: Let $R \subset \mathbb{C}$ be a region (not necessarily simply connected), let $f(z)$ be analytic in R, and let $F(z)$ be an indefinite integral of $f(z)$ that is single-valued $\forall z \in R$. Then

$$\int_{\mathcal{C}} dz\, f(z) = F(z_1) - F(z_0)$$

for all paths that start in z_0 and end in z_1.

remark: (4) If R is not simply connected, then we can always cut it such that $\mathcal{C}$ does not cross any cuts

→ We can assume $\int_{\mathcal{C}} dz\, f(z)$ to be single-valued

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proof: Let $\mathcal{C}(t_0) = z_0$, $\mathcal{C}(t) = z$ and define

$$F_1(z) = \int_{\mathcal{C}} dz'\, f(z')$$

Then
$$\frac{dF_1}{dz}(z) = \frac{d}{dz} \int_{t_0}^{t} dt'\, f(\mathcal{C}(t'))\, \frac{d\mathcal{C}}{dt'} = \frac{1}{\delta z} \int_{t}^{t+\delta t} dt'\, f(\mathcal{C}(t'))\, \frac{d\mathcal{C}}{dt'}$$

$$= \frac{1}{\delta z} \underbrace{\delta t \left.\frac{d\mathcal{C}}{dt'}\right|_t}_{=\delta z} f(\mathcal{C}(t)) = f(\mathcal{C}(t)) = f(z)$$

→ $F_1(z) = F(z) + \text{const}$. Choose $z = z_0$ → $F_1(z) = 0$
→ $\text{const} = -F(z_0)$

Week 8
§8 (26, 27, 28, 29, 30)

3.2 Laurent series

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lemma: Let $f: R \to \mathbb{C}$ be analytic, and let $z_0 \in R$. Then $\exists R > 0$:

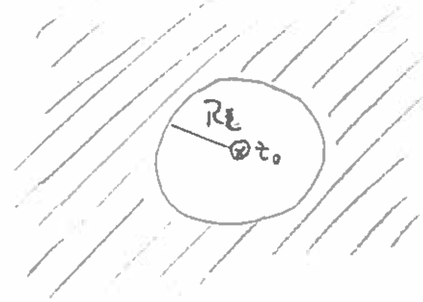
$$(*) \quad \boxed{f(z) = \sum_{n=0}^{\infty} f_n (z - z_0)^n} \quad \forall z \text{ with } |z - z_0| < R$$

This representation of f in the vicinity of z_0 is unique, and

$f_n = \frac{1}{n!} d^n f / dz^n \big|_{z_0}$ proof: E.g., Smirnow III/2 sec. 11

remark: (1) $(*)$ is called the Taylor series for f about z_0.

corollary 1: let $\tilde{g}(z) = \sum_{n=0}^{\infty} \tilde{g}_n (z-z_0)^n$ for $|z-z_0| < R_1$ be the Taylor for the fct. $\tilde{g}$ in the vicinity of z_0. Then $\exists R_2 > 0$: the fct. $g(z)$ defined by



$$g(z) := \sum_{n=0}^{\infty} \tilde{g}_n \frac{1}{(z-z_0)^n}$$

is analytic for all z with $|z-z_0| > R_2$.

proof: Define $\tilde{z} := \frac{1}{z-z_0} \rightsquigarrow g(z) = \sum_{n=0}^{\infty} \tilde{g}_n \tilde{z}^n =: \tilde{g}(\tilde{z})$.
lemma $\rightsquigarrow \exists 1/R_2 > 0$: $\tilde{g}(\tilde{z})$ exists and is analytic for $|\tilde{z}| < 1/R_2$
$\rightsquigarrow g(z)$ exists and is analytic for $\frac{1}{|z-z_0|} < R_2 \Leftrightarrow |z-z_0| > R_2$

corollary 2: The fct. $h(z) := f(z) + g(z) = \sum_{n=0}^{\infty} f_n (z-z_0)^n + \sum_{n=0}^{\infty} \tilde{g}_n (z-z_0)^{-n}$

$$= \sum_{n=-\infty}^{\infty} h_n (z-z_0)^n \quad \text{with} \quad h_n = \begin{cases} f_n & \text{for } n>0 \\ f_n + \tilde{g}_n & \text{for } n=0 \\ \tilde{g}_{-n} & \text{for } n<0 \end{cases}$$

is analytic for $R_2 < |z-z_0| < R_1$

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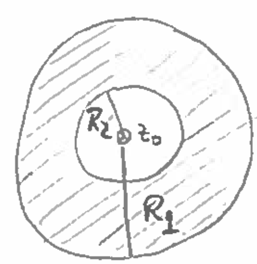
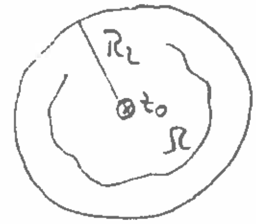
def. 1: $\sum_{n=-\infty}^{\infty} h_n (z-z_0)^n$ is called the Laurent series for the function $h(z)$ on the annulus $R_2 < |z-z_0| < R_1$.



theorem: let $f(z)$ be analytic on an annulus A around z_0. Then on A $f(z)$ can be uniquely expanded in a Laurent series, i.e., there exist unique coefficients f_n: $\boxed{f(z) = \sum_{n=-\infty}^{\infty} f_n (z-z_0)^n}$ $\forall z \in A$

proof: E.g., Smirnov III/2 No. 15

Now consider the



special case: $R_2 = 0$. $f(z)$ analytic in a region Ω except for $z = z_0$.

3 possibilities: (i) $f_n = 0 \ \forall n < 0$. Then $f(z)$ is analytic in z_0 as well

(ii) $\exists n<0 : f_n \neq 0 \wedge \exists m > 0 : f_n = 0 \ \forall n < -m$.

remark: (2) The strongest singularity is a term $\frac{1}{(z-z_0)^m}$

def. 2: We say that $f(z)$ has a pole of multiplicity m at the point z_0. For $m = 1$ we call the pole a simple pole.

(iii) $f_n \neq 0 \ \forall n < 0$.

remark: (3) The singularity is stronger than any inverse power.

def. 3: We say that $f(z)$ has an essential singularity at z_0.

def. 4: f_{-1} is called the residue of f in z_0, $f_{-1} =: \text{Res}\, f(z_0)$

example: (1) $f(z) = 1/z$ has a simple pole at $z = 0$, and $\text{Res}\, f(0) = 1$

(2) $f(z) = \frac{1}{(z-i)^2}$ has a pole of multiplicity 2 at $z = i$, and $\text{Res}\, f(i) = 0$.

(3) $f(z) = e^{-1/z}$ has an essential singularity at $z = 0$.

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(4) $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}$, $f(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ has $f_n = 0$ $\forall n < 0$ and can be analytically continued to all of $\mathbb{C}$: $\tilde{f} : \mathbb{C} \to \mathbb{C}$, $\tilde{f}(z) := \begin{cases} f(z) & \text{for } z \neq 0 \\ 1 & \text{for } z = 0 \end{cases}$

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is ... analytic $\forall z \in \mathbb{C}$.

3.3 The residue theorem

Lemma 1: Let $f(z) = (z-z_0)^n$ with $n \in \mathbb{Z}, n \neq -1$.

Let $\mathcal{C}$ be a closed path that goes around z_0.

Then $$\oint_{\mathcal{C}} dz\, f(z) = 0 .$$

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proof: For $n \geq 0$ this follows from the theorem in §2.1

For $n \leq -2$ we have an indefinite integral of f given by

$$F(z) = \frac{1}{n+1} (z-z_0)^{n+1}$$

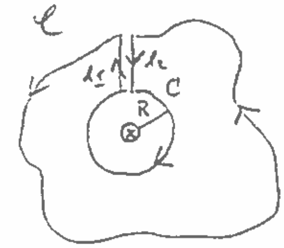
which is single-valued and differentiable everywhere except for $z = z_0$. §2.1 proposition $\Rightarrow$ $\oint_{\mathcal{C}} dz\, f(z) = \underline{F(z_1) - F(z_1)}_{=0}$ □

remark: (1) For $n = -1$ this argument breaks down since there is no single-valued fct. $F(z)$ such that $\frac{dF}{dz} = f(z)\ \forall z \neq z_0$!

Lemma 2. Let $f(z) = \frac{1}{z-z_0}$, and let $\mathcal{C}$ be a closed path that goes around z_0 counterclockwise. Then

$$\boxed{\oint_{\mathcal{C}} dz\, f(z) = 2\pi i}$$

proof: Consider the path $\mathcal{C} + l_1 + C + l_2$, which does not contain z_0.



$$\Rightarrow \oint_{\mathcal{C}+l_1+l_2+C} dz\, f(z) = 0$$

$$\| $$

$$\oint_{\mathcal{C}} dz\, f(z) + \oint_{C} dz\, f(z) + \overbrace{\int_{l_1} dz\, f(z) + \int_{l_2} dz\, f(z)}^{=0}$$

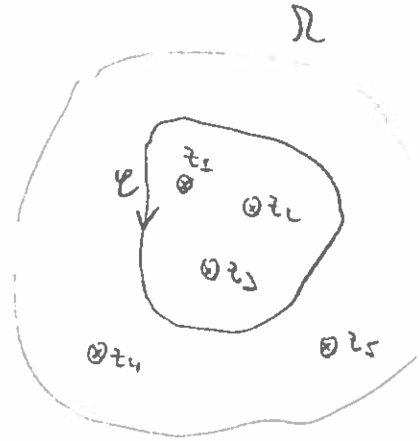
skip

$\to \oint_{\mathcal{C}} dz\, f(z) = -\oint_{C} dz\, f(z)$

But C we can parametrize by $z - z_0 = Re^{i\varphi}$ $(0 \le \varphi < 2\pi) \to dz = iRe^{i\varphi}$

$\to \oint_{C} dz\, f(z) = iR\int_0^{-2\pi} d\varphi\, e^{i\varphi}\frac{1}{Re^{i\varphi}} = -2\pi i \quad \to \quad \underline{\oint_{\mathcal{C}} dz\, f(z) = 2\pi i}$ □

Theorem: Residue theorem



Let $f: \mathcal{R} \to \mathbb{C}$ be analytic except for isolated points $z_1, z_2, \ldots \in \mathcal{R}$. Let $\mathcal{C}$ be a closed path that encloses $z_1, \ldots, z_n$. Then

$$\boxed{\oint_{\mathcal{C}} dz\, f(z) = 2\pi i \sum_{j=1}^{n} \mathrm{Res}\, f(z_j)}$$

when the integration along $\mathcal{C}$ is counterclockwise.

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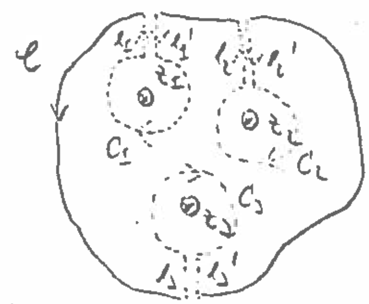
proof: Consider the contour



$$\mathcal{C} + \sum_{j=1}^{n}(l_j + l_j') + \sum_{j=1}^{n} C_j$$

along which the integral vanishes.

The integrations along the straight-line segments vanish independently, and we have

$$\oint_{\mathcal{C}} dz\, f(z) = -\sum_{j=1}^{n}\oint_{C_j} dz\, f(z)$$

But §22 theorem → the circles can be chosen such that $f(z)$ can be uniquely expanded in a Laurent series along the circle

$$\to \underbrace{\oint_{\mathcal{C}} dz\, f(z)}_{\text{lemma 2}} = -\sum_{j=1}^{n}\oint_{C_j} dz \sum_{m=-\infty}^{\infty} f_m^{(j)}(z-z_j)^m \overset{\text{lemma 1}}{=} -\sum_{j=1}^{n}\oint_{C_j} dz\, f_{-1}^{(j)}\frac{1}{z-z_j}$$

$$= \sum_{j=1}^{n} f_{-1}^{(j)} \cdot 2\pi i \sum_{j=1}^{n}\mathrm{Res}\, f(z_j)$$

remark: (1) This theorem is invaluable for many different physical applications

(2) We have reduced a class of integrals to determining the residues of the integrand!

proposition: Let $f(z)$ have a pole of multiplicity n in the point z_0. Then

$$\boxed{\operatorname{Res} f(z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\bigg|_{z=z_0} \left[f(z)(z-z_0)^n \right]}$$

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proof: Define $g(z) := (z-z_0)^n f(z)$, which is analytic in z_0.

§2.2 lemma $\rightarrow$ In the vicinity of z_0, $g(z)$ can be expanded

$$g(z) = g_0 + g_1(z-z_0) + g_2(z-z_0)^2 + \dots \quad \text{with } g_n = \frac{1}{n!} \frac{d^n}{dz^n}\bigg|_{z_0} g(z)$$

$$\rightarrow f(z) = \frac{g_0}{(z-z_0)^n} + \frac{g_1}{(z-z_0)^{n-1}} + \dots + \frac{g_{n-1}}{z-z_0} + g_n + \dots$$

$$\rightarrow \underline{\underline{\operatorname{Res} f(z_0)}} = g_{n-1} = \underline{\underline{\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\bigg|_{z_0} \left((z-z_0)^n f(z)\right)}} \quad \square$$

remark: (3) $\underline{n = 1}$: $\underline{\operatorname{Res} f(z_0)} = (z-z_0) f(z)\big|_{z=z_0} = \underline{f_{-1}}$ §2.2 def. 4 ✓

(4) $f(z) = \frac{1}{(z-z_0)^n}$, $n \neq 1$: $\underline{\operatorname{Res} f(z_0)} = \frac{d^{n-1}}{dz^{n-1}}\bigg|_{z_0} \frac{(z-z_0)^n}{(z-z_0)^n} = \underline{0}$

§2.3 lemma 1 ✓

(5) Strategy for doing integrals:

(i) determine the analytic structure of the integrand

(ii) deform the contour to simplify the integration while staying away from singularities (poles or branch cuts)

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3.4 Simple applications of the residue theorem

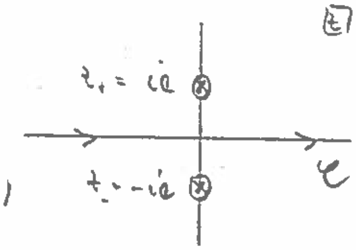
Consider some simple examples:

example: (1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x^2+a^2}$ $(a \in \mathbb{R})$

Find $g(\lambda) := \int_{-\infty}^{\infty} dx\, e^{-i\lambda x} f(x)$ "Fourier transform"

Continue f into the complex plane:

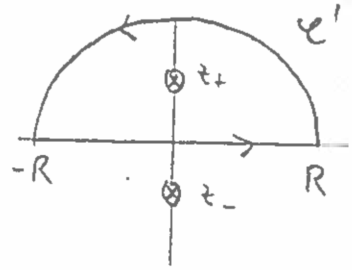
$$f(z) = \frac{1}{z^2+a^2} = \frac{1}{(z+ia)(z-ia)} =: \frac{1}{(z-z_-)(z-z_+)}$$



f is analytic everywhere except at $z_\pm = \pm ia$.

1st case: $\lambda < 0$

Consider $\tilde{g}(\lambda) := \int_{\mathcal{C}'} dz\, e^{-i\lambda z} f(z)$



$$= 2\pi i \operatorname{Res}\Big|_{z_+} f(z) e^{-i\lambda z}$$

$$= 2\pi i \frac{e^{-i\lambda z_+}}{z_+ - z_-} = 2\pi i \frac{e^{-i\lambda i a}}{2ia} = \frac{\pi}{a} e^{\lambda a} = \frac{\pi}{a} e^{-|\lambda| a}$$

and $\gamma_R(\lambda) := \int_{\frown} dz\, e^{-i\lambda z} f(z)$ (integration along semi-circle with radius R

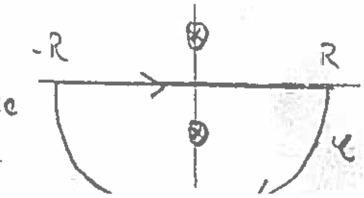
Then $g(\lambda) = \tilde{g}(\lambda) - \lim_{R\to\infty} \gamma_R(\lambda)$

But $\left|\lim_{R\to\infty} \gamma_R(\lambda)\right| = \left|\lim_{R\to\infty} \int_{\frown} dz\, e^{-i\lambda z} f(z)\right| \leq \lim_{R\to\infty} \pi R \left|e^{-i\lambda z} f(z)\right|$

$$\leq \lim_{R\to\infty} \frac{\pi R}{R^2+a^2} = 0 \quad \rightarrow g(\lambda) = \tilde{g}(\lambda)$$

2nd case: $\lambda > 0$

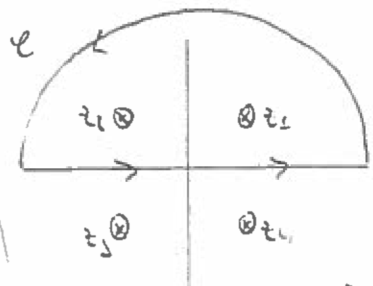
Same argument with $z_+ \to z_-$ $\rightarrow g(\lambda) = \frac{\pi}{a} e^{-\lambda a}$



$\rightarrow$ $g(\lambda) = \frac{\pi}{a} e^{-|\lambda| a}$

(2) $\mathcal{I} = \int_{-\infty}^{\infty} dx \, \frac{1}{x^4+1} = ?$

Consider $f(z) = \frac{1}{z^4+1}$. Poles: $z^4 = -1 \to z_1 = e^{i\pi/4}$, $z_2 = e^{3i\pi/4}$, $z_3 = e^{5i\pi/4}$, $z_4 = e^{7i\pi/4}$



$$\to \mathcal{I} = \int_{\mathcal{C}} dz\, f(z) = 2\pi i \left[\mathrm{Res}\, f(z_1) + \mathrm{Res}\, f(z_2)\right]$$

$$\mathrm{Res}\, f(z_1) = \left.\frac{z-z_1}{z^4+1}\right|_{z=z_1} = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{1}{(e^{i\pi/4}-e^{i3\pi/4})(e^{i\pi/4}-e^{i5\pi/4})(e^{i\pi/4}-e^{i7\pi/4})} = \frac{e^{-i\pi/4}}{(1-i)(1+1)(1+i)} = \frac{e^{-i\pi/4}}{4}$$

skip

$$\mathrm{Res}\, f(z_2) = \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{e^{-i3\pi/4}}{(1+i)(1-i)(1+1)} = \frac{1}{4} e^{-i3\pi/4}$$

$$\to \mathcal{I} = 2\pi i \, \frac{1}{4} e^{-i\pi/4}\left(1 + e^{-i\pi/2}\right) = \frac{i\pi}{2}\left(e^{-i\pi/4} + e^{-i3\pi/4}\right)$$

$$= \frac{i\pi}{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \frac{\pi}{2}\sqrt{2} = \frac{\pi}{\sqrt{2}}$$

Problem 3L

application of residue theorem

(3) $S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$ with $\Omega_n = 2\pi T n$ "Matsubara frequency"

Remark: (3) basis of this is an important ingredient in finite-T quantum field theory.

Let $f(z)$ have poles at z_j $(j = 1, 2, ..)$ and no other singularities

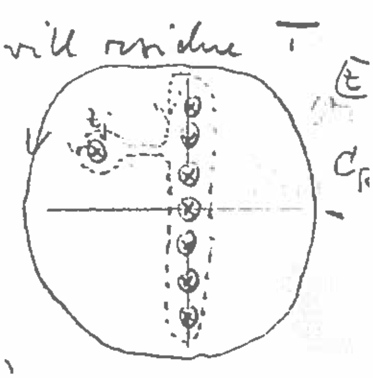
Let $f(|z| \to \infty)$ fall off faster than $1/z$.

Consider $n(z) = \frac{1}{e^{z/T}-1} \to n(i\Omega_n + \delta z) = \frac{1}{e^{2\pi i n + \delta z/T}-1} = \frac{T}{\delta z}$

$\to n(z)$ has simple poles at $z = i\Omega_n$ with residue T

Consider $\mathcal{I} = \lim_{R\to\infty} \oint_{C_R} \frac{dz}{2\pi i} f(z) n(z) = 0$

$$= T \sum_n f(i\Omega_n) + \sum_j \mathrm{Res}\, f(z_j)\, n(z_j)$$



$$\to S = \sum_j \mathrm{Res}\, f(z_j)\, n(z_j)$$ Wed 4/9

3.5 Another application of complex analysis: The Airy function $Ai(x)$

Consider the following 2nd order ODE for a real-valued fct y of a real variable x:

$$\boxed{\frac{d^2y}{dx^2} - xy = 0} \quad (*)$$

remark: (1) This ODE appears in QM (charged particle in an electric field) and in Optics (theory of the rainbow).

ansatz:

$$\boxed{y(x) = \int_a^b dt\, e^{xt} f(t)} \quad (**)$$

with $f(t)$ a fct. to be determined, and a and b constants that we are still free to choose.

remark: (2) For $a=0$, $b=\infty$ this is called the Laplace trafo of f

$$(**) \rightarrow \frac{d^2y}{dx^2} \equiv y''(x) = \int_a^b dt\, t^2 e^{xt} f(t)$$

$$\text{in } (*) \rightarrow \underline{0} = \int_a^b dt\, (t^2 - x) e^{xt} f(t) = \int_a^b dt \left(\left(t^2 - \frac{\partial}{\partial t} \right) e^{xt} \right) f(t)$$

$$\overset{\text{part. int.}}{=} \int_a^b dt\, t^2 e^{xt} f(t) + \int_a^b dt\, e^{xt} f'(t) - e^{xt} f(t) \Big|_a^b$$

$$= \underline{\int_a^b dt\, e^{xt} \left[t^2 f(t) + f'(t) \right] - e^{xt} f(t) \Big|_a^b}$$

$\rightarrow$ lemma: If f solves the ODE $\boxed{f'(t) + t^2 f(t) = 0}$ $(+)$

and $\boxed{e^{xt} f(t) \Big|_a^b = 0}$ $(++)$

then $y(x)$ as defined by $(**)$ solves the ODE $(*)$.

$(+)$ is separable: $\frac{df}{f} = -t^2 dt \rightarrow \log f = -\frac{1}{3} t^3 + \text{const}$

$\rightarrow \underline{f(t) = e^{-t^3/3}}$ is a solution of $(+$

(**) requires $e^{xt-t^3/3}\Big|_a^b = 0$

problem: $e^{xt-t^3/3} = 0$ only for $t = +\infty$!

solution: Generalize (**) by integrating in the complex plane:

proposition: $\boxed{y(x) = \text{const} \times \int_{\mathcal{C}} dt\, e^{tx - t^3/3}}$ (**')

solves (*) provided the integrand vanishes at the start and end points of the path $\mathcal{C}$.

-> It suffices to choose a path $\mathcal{C}$ that goes to ∞ in a region where $\mathrm{Re}\, z^3 > 0$

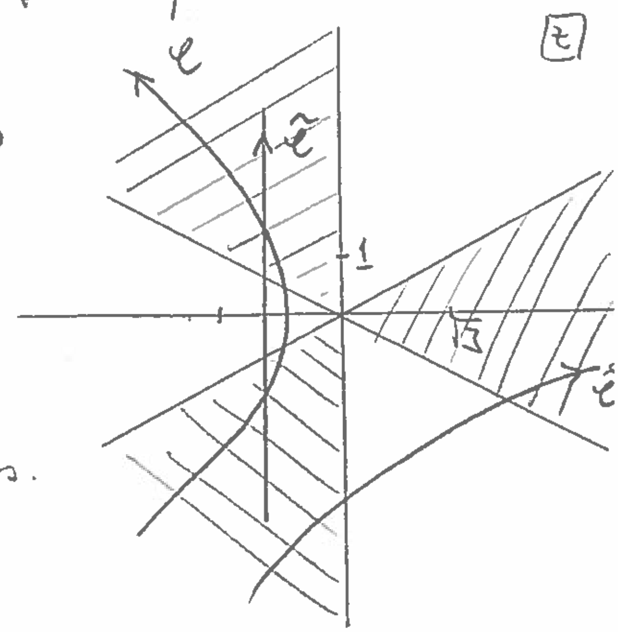
$\mathrm{Re}\, z^3 = \mathrm{Re}\,(z' + i z'')^3 = z'(z'^2 - 3z''^2) > 0$

$\Leftrightarrow$ either $z' > 0 \wedge z'^2 > 3z''^2$

or $z' < 0 \wedge z'^2 < 3z''^2$

-> $\mathcal{C}$ must go to ∞ in the shaded regions.

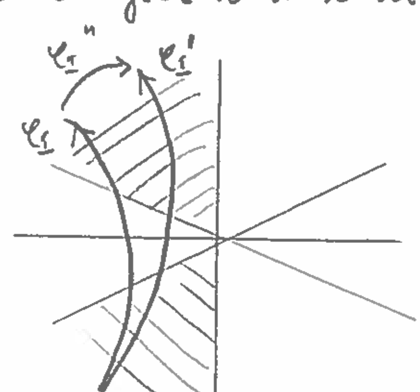
$\mathcal{C}, \hat{\mathcal{C}}, \hat{\hat{\mathcal{C}}}$ all qualify.



remark: (1) (**') is a solution of (*) for any $\mathcal{C}$ that goes to ∞ in the shaded regions.

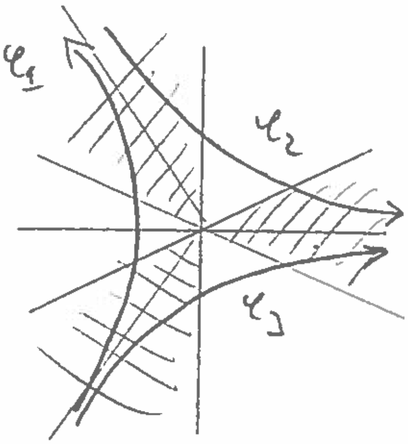
(2) The integrand is analytic

-> Paths that start in the same region and end in the same region lead to the same function. E.g.,



$$\Big(\int_{\mathcal{C}_1} dt + \underbrace{\int_{\mathcal{C}_1''} dt}_{=0} - \int_{\mathcal{C}_1'} dt \Big) e^{xt - t^3/3} = 0 \quad \text{by §3.1 corollary}$$

$$\to \int_{\mathcal{C}_1} dt\, e^{xt-t^3/3} = \int_{\mathcal{C}_1'} dt\, e^{xt-t^3/3}$$



(5) This implies that there are only three essentially different paths which end at two of the three points $+\infty$, $\infty e^{2\pi i/3}$, $\infty e^{-2\pi i/3}$

Then §3.1 corollary →

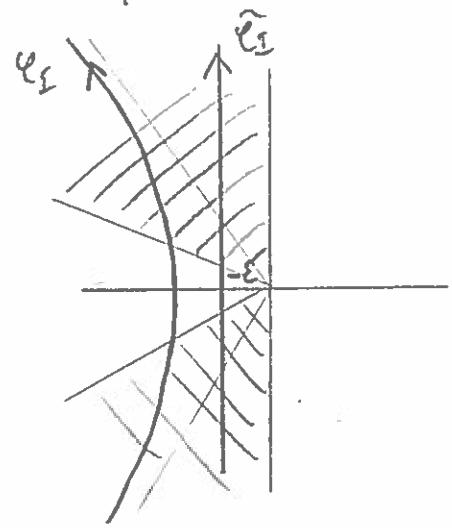
$$\left(\int_{\mathcal{C}_1} dz + \int_{\mathcal{C}_2} dz - \int_{\mathcal{C}_3} dz \right) e^{xz - z^3/3} = 0$$

→ Of the three paths, only two are independent (as expected since there integrals are solutions of a 2nd order ODE.

11/16/16

• 3.5.1 The Airy function Ai(x)

Define
$$\boxed{Ai(x) := \frac{1}{2\pi i} \int_{\mathcal{C}_1} dz \, e^{xz - z^3/3}} \qquad (*)$$
Airy function



Alternatively, deform the contour from $\mathcal{C}_1$ to $\mathcal{C}_1'$

$$\to \quad Ai(x) = \frac{1}{2\pi i} \int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} dz \, e^{xz - z^3/3} = \begin{bmatrix} z = iu, \; u = -iz \\ dz = i\,du \end{bmatrix}$$

$$= \frac{1}{2\pi} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} du \, e^{iux + iu^3/3}$$

$$\overset{\varepsilon \to 0}{=} \frac{1}{2\pi} \int_{-\infty}^{\infty} du \left[\cos(ux + u^3/3) + \underbrace{i \sin(ux + u^3/3)}_{\to 0 \text{ by symmetry}} \right]$$

$$\to \quad \boxed{Ai(x) = \frac{1}{\pi} \int_{0}^{\infty} du \, \cos(ux + u^3/3)} \qquad (**)$$

alternative integral representation for Ai(x) remark: (1) Ai(x) is finite for all x (see below). The second independent solution ($\mathcal{C}_2$, or $\mathcal{C}_3$, or a linear combination of the two) results in a function Bi(x) that goes to ∞ for $x \to \infty$ (see Bender & Orszag or Stone & Goldbart)

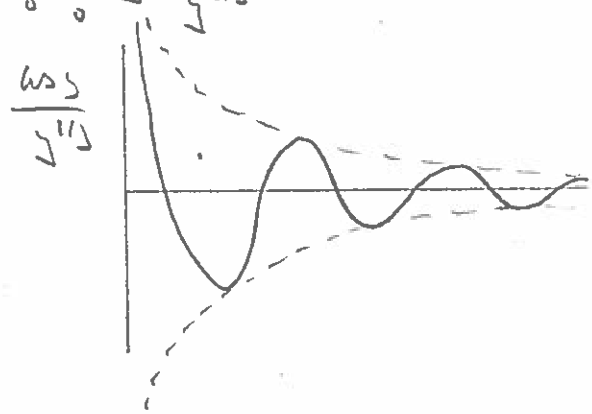
(2) The existence of the integral in (**) is not obvious, since the integrand does not fall off as $u \to \infty$. Convergence is assured by the equivalence of (**) and (*).

Discussion of Ai(x):

$x = 0$: $\quad Ai(x=0) = \frac{1}{\pi}\int_0^\infty du \cos\left(\frac{1}{3}u^3\right) = \begin{bmatrix} u^3 = y, & u = y^{1/3} \\ dy = 3u^2 du, & du = dy/3y^{2/3} \end{bmatrix}$

$$= \frac{1}{3\pi}\int_0^\infty dy \frac{\cos(y/3)}{y^{2/3}} = \frac{1}{3^{2/3}\pi}\int_0^\infty dy \frac{\cos y}{y^{2/3}}$$

singularity at $y=0$ is integrable
oscillations lead to convergence for $y \to \infty$



$$= \frac{1}{3^{2/3}\,\Gamma(2/3)} = 0.355$$

$x \to \pm\infty$: Method of steepest descent

$$\int_{\mathcal{C}} dz\, e^{f(z)} = \int_{\mathcal{C}} dz\, e^{f(z_0) + \frac{1}{2}(z-z_0)^2 f''(z_0) + \dots}$$ with z_0 a maximum of $f(z)$ along $\mathcal{C}$

$$\simeq e^{f(z_0)} \int_{\mathcal{C}} dz\, e^{\frac{1}{2}(z-z_0)^2 f''(z_0)}$$

here: $\quad f(z) = xz - \frac{1}{3}z^3$

$\rightarrow 0 \overset{!}{=} f'(t_0) = x - t_0^2 \rightarrow \underline{t_0 = \pm\sqrt{x}}$

1st case: $\underline{x > 0}$ let $t = \pm\sqrt{x} + iu$

$\rightarrow \underline{f(t)} = x(\pm\sqrt{x} + iu) - \frac{1}{3}(\pm\sqrt{x} + iu)^3$

$= \pm \cancel{x}^{3/2} + \cancel{ixu} \mp \frac{1}{\cancel{3}}\cancel{x}^{3/2} - \cancel{ixu} \pm \sqrt{x}u^2 + \frac{i}{3}u^3$

$= \underline{\pm \frac{2}{3}x^{3/2} \pm \sqrt{x}u^2 + O(u^3)}$

$\rightarrow$ For a vertical contour, $t_0 = -\sqrt{x}$ is a maximum

$t_0 = \sqrt{x}$ is a minimum

$$\rightarrow \underline{\underline{Ai(x \to \infty)}} \simeq \frac{i}{2\pi i} e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} du\, e^{-\sqrt{x}u^2} = \frac{1}{2\pi} \frac{e^{-2x^{3/2}/3}}{x^{1/4}} \underbrace{\int_{-\infty}^{\infty} du\, e^{-u^2}}_{=\sqrt{\pi}}$$

$$= \underline{\underline{\frac{1}{2\sqrt{\pi}} \frac{1}{x^{1/4}} e^{-\frac{2}{3}x^{3/2}}}}$$

remark: (3) Consider corrections to this result: The full u-integral is

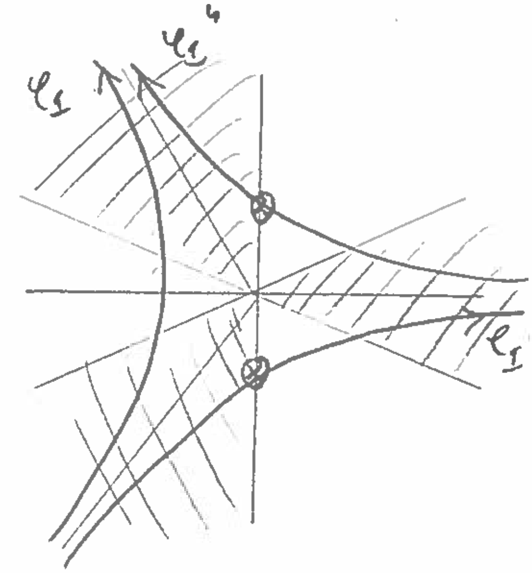
$$\underline{\int_{-\infty}^{\infty} du\, e^{-\sqrt{x}u^2 + \frac{i}{3}u^3}} = \frac{1}{x^{1/4}} \int_{-\infty}^{\infty} du\, e^{-u^2} e^{\frac{i}{3}u^3/x^{3/4}}$$

$$= \frac{1}{x^{1/4}} \int_{-\infty}^{\infty} du\, e^{-u^2} \left[1 + \frac{i}{3}\frac{u^3}{x^{3/4}} - \frac{1}{18}\frac{u^6}{x^{3/2}} + \dots\right]$$

$$= \underline{\frac{\sqrt{\pi}}{x^{1/4}} \left[1 + O(x^{-3/2})\right]}$$ this really is an expansion for $x \gg 1$!

2nd case: $\underline{x < 0}$ $\rightarrow$ Two saddle points at $t = \pm i\sqrt{|x|}$

Deform the contour $\mathcal{C}_1 \rightarrow \mathcal{C}_1' + \mathcal{C}_1''$ such that it goes through both saddle points:



let $z = i\sqrt{|x|} + \xi$

and $\xi = e^{3\pi i/4} u$

$$\rightarrow f(z) = x\left(i\sqrt{|x|} + \xi\right) - \tfrac{1}{3}\left(i\sqrt{|x|} + \xi\right)^3$$

$$= \cancel{-i|x|^{3/2}} \cancel{- |x|\xi}$$

$$- \tfrac{1}{3}\left(\cancel{-i|x|^{3/2}} \cancel{- 3|x|\xi} + 3i|x|^{1/2}\xi^2\right) + O(\xi^3)$$

$$= -\tfrac{2}{3} i|x|^{3/2} - i|x|^{1/2}\xi^2 + O(\xi^3)$$

$$= -\tfrac{2}{3} i|x|^{3/2} - |x|^{1/2}u^2 + O(u^3)$$

$$\rightarrow \int_{\mathcal{C}_1''} dz\, e^{f(z)} \simeq e^{-\frac{2}{3} i|x|^{3/2}} \int_{-\infty}^{\infty} du\, e^{-\sqrt{|x|}u^2} e^{3\pi i/4}$$

Similarly, let $z = -i\sqrt{|x|} + \xi$ and $\xi = e^{\pi i/4} u$

$$\rightarrow \int_{\mathcal{C}_1'} dz\, e^{f(z)} \simeq e^{\frac{2}{3} i|x|^{3/2}} \int_{-\infty}^{\infty} du\, e^{-\sqrt{|x|}u^2} e^{\pi i/4}$$

$$\rightarrow \underline{Ai(x\rightarrow -\infty)} \simeq \frac{1}{2\pi i}\left[e^{\frac{2}{3} i|x|^{3/2}} e^{\pi i/4} + e^{-\frac{2}{3} i|x|^{3/2}} e^{3\pi i/4}\right] \underbrace{\int_{-\infty}^{\infty} du\, e^{-\sqrt{|x|}u^2}}_{=\frac{1}{|x|^{1/4}}\sqrt{\pi}}$$

$$= \frac{\sqrt{\pi}}{2\pi i}\frac{1}{|x|^{1/4}}\left[e^{\frac{2}{3} i|x|^{3/2}} e^{-i\frac{\pi}{4}} + e^{-\frac{2}{3} i|x|^{3/2}} e^{i\frac{\pi}{4}}\right] \underbrace{e^{i\frac{\pi}{2}}}_{=i}$$

$$= \frac{1}{2\sqrt{\pi}|x|^{1/4}}\, 2\cos\left(\tfrac{2}{3}|x|^{3/2} - \tfrac{\pi}{4}\right)$$

$$= \underline{\underline{\frac{1}{\sqrt{\pi}|x|^{1/4}}\sin\left(\tfrac{2}{3}|x|^{3/2} + \tfrac{\pi}{4}\right)}}$$

Conclusion: $Ai(x)$ decays exponentially with a power-law correction for $x\rightarrow\infty$, and is oscillatory with a $1/|x|^{1/4}$ envelope for $x\rightarrow -\infty$

71g

In[1]:= Plot[AiryAi[x], {x, -10, 10}]

Out[1]=