§9 Fourier transform and generalized functions (see Lighthill ch. 2)

9.1 The Fourier transform in classical analysis

Let $f: \mathbb{R}^n \to \mathbb{C}$ be a complex-valued fct. that is absolutely integrable, i.e., $\int d\vec{x}\, |f(\vec{x})| < \infty$

remark: (0) The space $\mathcal{Y}^{(1)}$ of such fct is a vector space over $\mathbb{C}$ and addition of fcts. (proof easy)

notation: $\vec{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\int d\vec{x} = \int_{\mathbb{R}^n} dx_1 \dots dx_n$

$\vec{\lambda}\cdot\vec{x} = \lambda_1 x_1 + \dots + \lambda_n x_n$ with $\vec{\lambda} \in \mathbb{R}^n$.

def. 1: $\boxed{\hat{f}(\vec{\lambda}) := \int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} f(\vec{x})} \equiv FT(f)(\vec{\lambda})$

tables 32, 33 Fourier trafos

is called Fourier transform of $f(\vec{x})$

table 34 von FTs Kronecker-
Wong

remark: (1) $\hat{f}: \mathbb{R}^n \to \mathbb{C}$ is another complex-valued fct. on $\mathbb{R}^n$.

(2) The FT is a linear integral trafo.

(3) $FT(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 FT(f_1) + \lambda_2 FT(f_2)$ $\forall \lambda_{1,2} \in \mathbb{C}$ due to linearity.

proposition 1: $\hat{f}(\vec{\lambda})$ is bounded and continuous

proof: $|\hat{f}(\vec{\lambda})| = \left|\int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} f(\vec{x})\right| \le \int d\vec{x}\, \left|e^{-i\vec{\lambda}\vec{x}} f(\vec{x})\right| = \int d\vec{x}\, |f(\vec{x})| < \infty$

$|\hat{f}(\vec{\lambda}_1) - \hat{f}(\vec{\lambda}_2)| = \left|\int d\vec{x}\,(e^{-i\vec{\lambda}_1\vec{x}} - e^{-i\vec{\lambda}_2\vec{x}}) f(\vec{x})\right| \le$

$\le \int d\vec{x}\, \left|e^{-i\vec{\lambda}_1\vec{x}} - e^{-i\vec{\lambda}_2\vec{x}}\right| |f(\vec{x})| \to 0$ for $\vec{\lambda}_1 \to \vec{\lambda}_2$

proposition 2: Let $x_\ell f(\vec{x})$ be absolutely integrable. Then $\hat{f}(\vec{\lambda})$ is differentiable with respect to λ_ℓ and

$$\frac{\partial}{\partial \lambda_\ell} \hat{f}(\vec{\lambda}) = FT(-ix_\ell f)(\vec{\lambda})$$

proof: $\frac{\partial}{\partial \lambda_\ell} \int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} f(\vec{x}) = -i \int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} x_\ell f(\vec{x}) = FT(-ix_\ell f)(\vec{\lambda})$

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proposition 3: Let $f(\vec{x})$ be differentiable with respect to x_ℓ and let $\frac{\partial}{\partial x_\ell} f$ be absolutely integrable. Then

$$FT(\partial_\ell f)(\vec{\lambda}) = +i\lambda_\ell \hat{f}(\vec{\lambda})$$

proof: (for $n=1$:)

$$\underline{\int dx\, e^{-i\lambda x} \frac{d}{dx} f(x)} \overset{\text{part. int.}}{=} -\int dx\, (-i\lambda)\, e^{-i\lambda x} f(x) - \underbrace{e^{-i\lambda x} f(x)\Big|_{-\infty}^{\infty}}_{=0}$$

$$= \underline{i\lambda \hat{f}(\lambda)}$$

remark: (4) FT turns derivatives into products. Prospect: Turn differential eqs into algebraic ones!

(5) This also works for $n>1$ and higher derivatives. For instance, for $n=3$, $FT(\nabla^2 f)(\vec{\lambda}) = -\vec{\lambda}^2 \hat{f}(\vec{\lambda})$.

proposition 4: $FT(f^*)(\vec{\lambda}) = \left(FT(f)(-\vec{\lambda})\right)^*$

proof: $\underline{\int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} (f(\vec{x}))^*} = \left(\int d\vec{x}\, e^{i\vec{\lambda}\vec{x}} f(\vec{x})\right)^* = \underline{\left(\hat{f}(-\vec{\lambda})\right)^*}$ □

theorem: Convolution theorem

Let f_1, f_2 be absolutely integrable, and let

$$f(\vec{y}) = \int d\vec{x}\, f_1(\vec{y}-\vec{x})\, f_2(\vec{x})$$

assumed to be absolutely integrable. Then

$$\boxed{\hat{f}(\vec{\lambda}) = \hat{f}_1(\vec{\lambda})\, \hat{f}_2(\vec{\lambda})}$$

proof: $\underline{\hat{f}(\vec{\lambda})} = \int d\vec{y}\, e^{-i\vec{\lambda}\vec{y}} \int d\vec{x}\, f_1(\vec{y}-\vec{x})\, f_2(\vec{x}) = \int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} \int d\vec{y}\, e^{-i\vec{\lambda}(\vec{y}-\vec{x})} f_1(\vec{y}-\vec{x})\, f_2(\vec{x})$

$$= \int d\vec{x}\, e^{-i\vec{\lambda}\vec{x}} f_2(\vec{x}) \int d\vec{z}\, e^{-i\vec{\lambda}\vec{z}} f_1(\vec{z})$$

$$= \underline{\hat{f}_2(\vec{\lambda})\, \hat{f}_1(\vec{\lambda})}$$ □

remark: (6) Convolutions in real space turn into products in Fourier space

9.2 Inverse Fourier transform

Let f_1, f_2 be absolutely integrable.

Lemma:
$$\boxed{\int d\vec{x}\; f_1(\vec{x})\,\big(\hat{f}_2(\vec{x})\big)^* = \int d\vec{y}\; \hat{f}_1(-\vec{y})\,\big(f_2(\vec{y})\big)^*}$$

proof:
$$\int d\vec{x}\, f_1(\vec{x})\left(\int d\vec{y}\, e^{-i\vec{y}\vec{x}} f_2(\vec{y})\right)^* = \int d\vec{x}\, f_1(\vec{x}) \int d\vec{y}\, e^{i\vec{y}\vec{x}} \big(f_2(\vec{y})\big)^*$$
$$= \int d\vec{y}\, \big(f_2(\vec{y})\big)^* \int d\vec{x}\, e^{i\vec{x}\vec{y}} f_1(\vec{x}) = \int d\vec{y}\, \hat{f}_1(-\vec{y})\,\big(f_2(\vec{y})\big)^* \qquad \square$$

Theorem: Let $f(\vec{x})$ and $\hat{f}(\vec{k})$ exist and be absolutely integrable. Then
$$\boxed{f(\vec{x}) = \frac{1}{(2\pi)^n} \int d\vec{k}\, e^{i\vec{k}\vec{x}} \hat{f}(\vec{k})}$$

Remark: (1) This means $FT(FT(f)) = (2\pi)^n f$, i.e., the FT is almost its own inverse!

proof. Consider the lemma with $f_1 = f$ and $f_2(\vec{y}) = e^{-\alpha \vec{y}^2} e^{i\vec{y}\cdot\vec{x}}$ $(\alpha > 0,\ \vec{x} \in \mathbb{R})$

$$\rightarrow \hat{f}_2(\vec{k}) = \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} e^{-\alpha\vec{y}^2} e^{i\vec{x}\cdot\vec{y}} = \int d\vec{y}\, e^{-i\vec{y}(\vec{k}-\vec{x})} e^{-\alpha \vec{y}^2}$$

Problem 27
$$= \left(\frac{\pi}{\alpha}\right)^{n/2} e^{-(\vec{k}-\vec{x})^2/4\alpha}$$

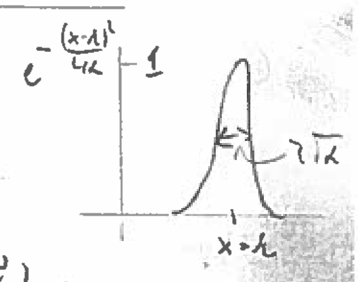
Lemma $\leadsto$
$$\int d\vec{y}\, \hat{f}(-\vec{y})\, e^{-\alpha\vec{y}^2} e^{-i\vec{y}\cdot\vec{x}} = \int d\vec{k}\, f(\vec{k}) \left(\frac{\pi}{\alpha}\right)^{n/2} e^{-(\vec{x}-\vec{k})^2/4\alpha}$$

Now consider the limit $\alpha \to 0$. Then

l.h.s. $\rightarrow \int d\vec{y}\, \hat{f}(-\vec{y})\, e^{-i\vec{y}\cdot\vec{x}} = \int d\vec{k}\, \hat{f}(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$

and by the intermediate value theorem

r.h.s. $\rightarrow f(\vec{x}) \left(\frac{\pi}{\alpha}\right)^{n/2} \int d\vec{k}\, e^{-(\vec{x}-\vec{k})^2/4\alpha}$



$$= f(\vec{x}) \left(\frac{\pi}{\alpha}\right)^{n/2} \left(\int dk\, e^{-k^2/4\alpha}\right)^n$$
$$= f(\vec{x}) \left(\frac{\pi}{\alpha}\right)^{n/2} (2\sqrt{\alpha})^n (\sqrt{\pi})^n = (2\pi)^n f(\vec{x}) \qquad \square$$

4.3 Test functions

problem: Even very simple fcts do not have Fourier transforms, e.g., $f(x) \equiv 1$.

solution: "Generalized functions" (see, e.g. the book by Lighthill)

In order to define generalized fcts we consider function spaces in addition to'

def. 1: A function $F: \mathbb{R} \to \mathbb{C}$ is called a test function if F is differentiable arbitrarily many times and if F and all of its derivatives go to zero faster than any power for $|x| \to \infty$.

example: (1) $F(x) = e^{-x^2}$ is a test function. So is $x^n e^{-mx^2}$ for all positive integers n, m.

def. 2: A function $\varphi: \mathbb{R} \to \mathbb{C}$ is called a weakly increasing function if φ is diff'able arbitrarily many times and if φ and all of its derivatives do not grow faster than $|x|^N$ for $|x| \to \infty$, where $N \in \mathbb{N}$ may depend on the order of the derivative.

example: (2) Any polynomial is a weakly increasing fct.

remark: (1) The derivative of a test function is a test function. So is the sum of two test fcts, and $\lambda F(x)$ with $\lambda \in \mathbb{R}$.
$\Rightarrow$ The set of test fcts forms a vector space.

(2) Let F be a test function and φ a weakly increasing fct. Then $G(x) := F(x)\varphi(x)$ is a test fct.

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proposition 1: If $F(x)$ is a test fct., then so is its Fourier transform

$$\hat{F}(\lambda) = \int dx\, e^{-i\lambda x} F(x)$$

proof: The p^{th} derivative of $\hat{F}(\lambda)$, $\hat{F}^{(p)}(\lambda) = \frac{d^p}{d\lambda^p} \hat{F}(\lambda) = (-i)^p \int dx\, x^p e^{[illegible]}$ [illegible]

obviously exists and [illegible]

$$|\hat{F}^{(p)}(\lambda)| = \left|\int dx\, x^p e^{-i\lambda x} F(x)\right| = \left|\int dx\, x^p F(x) \frac{1}{-i\lambda}\frac{d}{dx} e^{-i\lambda x}\right|$$

$$\overset{\text{part. int}}{=} \left|\frac{1}{-i\lambda}\int dx\, e^{-i\lambda x} \frac{d}{dx} x^p F(x)\right| \quad \text{(boundary terms vanish since } F \text{ is a test fct.)}$$

$$\overset{N-1 \text{ part. int.}}{=} \left|\frac{1}{(-i\lambda)^N}\int dx\, e^{-i\lambda x} \frac{d^N}{dx^N} x^p F(x)\right|$$

$$\leq \frac{1}{|\lambda|^N} \underbrace{\int dx \left|\frac{d^N}{dx^N} x^p F(x)\right|}_{<\infty} = O(|\lambda|^{-N}) \quad \forall N \in \mathbb{N}$$

□

remark: (1) The inverse Fourier transfo is given by the theorem from §4.

proposition 2: Parseval's equation

let $F_1(x)$ and $F_2(x)$ be test fcts, and $\hat{F}_1(\lambda), \hat{F}_2(\lambda)$ their Fourier transfos. Then

$$\boxed{\int \frac{d\lambda}{2\pi} \hat{F}_1(\lambda)\hat{F}_2(\lambda) = \int dx\, F_1(-x) F_2(x)}$$

proof: $\int d\lambda\, \hat{F}_1(\lambda)\hat{F}_2(\lambda) = \int d\lambda \int dx\, e^{-i\lambda x} F_1(x) \hat{F}_2(\lambda)$

$= \int dx\, F_1(x)\, 2\pi\, F_2(-x) =$ □

4.4 Generalized functions

def 1: let $f_n(x)$ $(n=1,2,..)$ be a sequence of test fcts. The sequence is called regular if $\lim_{n\to\infty} \int dx\, f_n(x) F(x)$ (*) exists for all test fcts $F(x)$.

remark: (1) The integral exists for all n, so the only issue is whether the limit exists.

example: (1) The sequence $\{ f_n(x) = e^{-x^2/n^2} ; n \in \mathbb{N} \}$ is regular, and

$$\lim_{n\to\infty} \int dx\, f_n(x) F(x) = \int dx\, F(x) \quad \forall F \in \mathcal{S}$$

proof: Problem #35

def. 2: Two regular sequences of test fcts are called equivalent if they yield the same limit (*).

example: (2) e^{-x^2/n^4} is equivalent to e^{-x^2/n^2}. So is $e^{-x^2/n}$.

def. 3: The set of all equivalent regular sequences $f_n(x)$ defines a generalized function† $f(x)$, and the integral

† or distribution

$$\int dx\, f(x) F(x) := \lim_{n\to\infty} \int dx\, f_n(x) F(x)$$

is defined by the limit on the r.h.s., which exists for all $F \in \mathcal{S}$ and is the same for all of the equivalent sequences.

Any of the equivalent sequences is called a regularization of the generalized fct. $f(x)$.

example: (3) $f_n(x) = e^{-x^2/n^2}$ and its equivalent sequences define the generalized fct. $f(x) \equiv 1$, and $f_n(x)$ is a regularization of $f(x)$.

remark: (2) The properties of the generalized fct $f(x) \equiv 1$ coincide with those of the ordinary fct. $f(x) \equiv 1$

remark: (3) Differentiation, addition, multiplication with smooth increasing fcts, and Fourier transform of generalized fcts can all be defined in terms of their representations and yield again generalized fcts. However, multiplication of generalized fcts with generalized fcts can NOT be consistently defined.

proposition 1: Let $f(x)$ be a fct. in the usual sense and that there exists an $N \in \mathbb{N}$ such that $f(x)/((1+x^2)^N)$ is absolutely integrable. Then one can construct sequences of test fcts $f_L(x)$ such that

$$\lim_{L\to\infty} \int dx\, f_L(x)\, F(x) = \int dx\, f(x)\, F(x)$$

for all test fcts F.

proof: see books (e.g., Lighthill ch. 2.3).

remark: (4) This says that a large class of ordinary fcts can be considered generalized fcts.

example: (4) Consider the ordinary fct. $\mathrm{sgn}(x) = |x|/x$, which fulfills the premises of proposition 1 with $N=1$. $\mathrm{sgn}\, x$ thus is a generalized fct, and $f_L(x) = \tanh(Lx)$ is a representation. (Proof of the latter: Problem 35)

Problem 35
representation of step and sgn

remark: (5) Such constructed generalized fcts are called regular. The derivative of any regular generalized fct. exists but it is in general not regular.

Problem 36
[illegible] limits

example: (5) $\frac{d}{dx}\,\mathrm{sgn}\, x$ is not regular, see Problem 36.

def. 4: Let $f_t(x)$ be a generalized fct for any value of the parameter t, and let $f(x)$ be another generalized fct. and that

$$\lim_{t\to c}\int dx\, f_t(x)F(x) = \int dx\, f(x)F(x)$$

for all test fcts F, where c may be finite or infinite and the set of parameters may be continuous or discrete. Then we say

$$\lim_{t\to c} f_t(x) = f(x)$$

example: (6) $\lim_{\varepsilon\to 0}|x|^{\varepsilon}\,\operatorname{sgn} x = \operatorname{sgn} x$

remark: (6) This is sometimes called a distribution limit

(7) Consider the test fcts $f_n(x)$ that make up a regular sequence in the sense of def. 1 generalized fcts, and let $f(x)$ be the generalized fct that is defined by this sequence and its equivalence class. Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

prop. 2: Under the conditions of def. 4 we have

(i) $\lim_{t\to c} f_t'(x) = f'(x)$

(ii) $\lim_{t\to c} f_t(ax+b) = f(ax+b)$

(iii) $\lim_{t\to c} \varphi(x)f_t(x) = \varphi(x)f(x)$ for any slowly increasing fct. $\varphi(x)$

proof: Trivial

4.5 The δ-function

def. 1: The generalized fct. $\delta(x)$ is defined as the set of equivalent regular sequences for which

$$\boxed{\int dx\, \delta(x) F(x) = \lim_{n\to\infty} \int dx\, f_n(x) F(x) = F(0)} \quad \forall F \in \mathcal{S}$$

remark: (1) There is no ordinary fct with this property.

proposition 1: $f_n(x) = \sqrt{n/\pi}\, e^{-nx^2}$ $(n \in \mathbb{N})$ is a regularization of $\delta(x)$

proof: $\int dx\, f_n(x) = \sqrt{\frac{n}{\pi}} \int dx\, e^{-nx^2} = \frac{1}{\sqrt{\pi}} \int dx\, e^{-x^2} = 1$

$$\to \left| \int dx\, f_n(x) F(x) - F(0) \right| = \left| \int dx\, f_n(x) \left(F(x) - F(0)\right) \right| \overset{\text{triangle ineq.}}{\leq}$$

$$\leq \int dx\, f_n(x) \left|F(x) - F(0)\right| = \int dx\, f_n(x) |x| \left| \frac{F(x) - F(0)}{x} \right|$$

$$\overset{F' \text{ bounded}}{\leq} (\sup F') \int dx\, |x| f_n(x) = 2 (\sup F') \int_0^\infty dx\, x \sqrt{\frac{n}{\pi}} e^{-nx^2}$$

$$= \frac{2}{\sqrt{\pi n}} (\sup F') \int_0^\infty dx\, x e^{-x^2} \to 0 \text{ for } n \to \infty \qquad \square$$

proposition 2: The Fourier transf of $\delta(x)$ is $\boxed{\hat{\delta}(k) = 1}$

proof: consider the regularization $f_n(x) = \sqrt{n/\pi}\, e^{-nx^2}$

Problem 32 $\to$ $\hat{f}_n(k) = e^{-k^2/4n}$

[§4.4 example 3] $\to$ This is a regularization of the generalized fct that's identically equal to 1.

corollary: The δ-fct. can be written $\delta(x) = \int \frac{d\lambda}{2\pi}\, e^{i\lambda x}$

proof: §4.2 Theorem $\Rightarrow$ $\underline{\delta(x)} = \int \frac{d\lambda}{2\pi}\, e^{i\lambda x}\, \hat{\delta}(\lambda) = \underline{\int \frac{d\lambda}{2\pi}\, e^{i\lambda x}}$ □

remark: (2) This integral does not exist in classical analysis!

proposition 3: Let $\varphi(x)$ be a slowly increasing fct. Then

$$\boxed{\varphi(x)\,\delta(x) = \varphi(0)\,\delta(x)}$$

proof: $\int dx\, \delta(x) \underbrace{\varphi(x) F(x)}_{=\text{test fct.}} = \varphi(0) F(0) = \varphi(0) \int dx\, \delta(x) F(x) \quad \forall F \in$ □

corollary: $\int dx\, \delta(x)\, \varphi(x) = \varphi(0)$

proof: $\underline{\int dx\, \delta(x)\, \varphi(x)} = \varphi(0) \int dx\, \delta(x) = \varphi(0)\, \hat{\delta}(\lambda = 0) = \underline{\varphi(0)}$ □

remark: (3) This is consistent with $\hat{\delta}(\lambda) = \int dx\, e^{-i\lambda x}\, \delta(x) = 1$,

(4) Even and odd generalized fcts can be defined in analogy to even and odd ordinary fcts.

example 1: $\delta(x) = \delta(-x)$ is even, since $\delta(-x) = \int \frac{d\lambda}{2\pi}\, e^{-i\lambda x} \overset{\lambda \to -\lambda}{=} \int \frac{d\lambda}{2\pi}\, e^{i\lambda x} = \delta(x)$

Accordingly, $\delta'(x) = \frac{d}{dx}\delta(x)$ is odd.

remark: (5) The δ-fct. makes the Fourier back transform easy:

$$\underline{\int \frac{d\lambda}{2\pi}\, \hat{f}(\lambda)\, e^{i\lambda x}} = \int \frac{d\lambda}{2\pi}\, e^{i\lambda x} \int dy\, e^{-i\lambda y} f(y) = \int dy\, \delta(y - x) f(y) = \underline{f(x)}$$

proposition 4: The δ-fct. has the properties

$$\text{(i)}\quad \delta(ax) = \frac{1}{|a|}\,\delta(x) \qquad \forall\, a \in \mathbb{R}\setminus\{0\}$$

$$\text{(ii)}\quad \varphi(x)\,\delta(a-x) = \varphi(a)\,\delta(a-x)$$

$$\text{(iii)}\quad \delta(\varphi(x)) = \sum_i \frac{1}{|\varphi'(x_i)|}\,\delta(x-x_i)$$ where the x_i are all real zeros of $\varphi(x)$ and we assume that the real zeros are simple and isolated.

proof: (i) $$\int dx\, F(x)\,\delta(ax) = \operatorname{sgn} a \int \frac{dx}{a} F(x/a)\,\delta(x) = \frac{\operatorname{sgn} a}{a} F(0) = \frac{1}{|a|} F(0) = \frac{1}{|a|}\int dx\, F(x)\,\delta(x) \qquad \forall F$$

(ii) $$\int dx\, F(x) f(x)\,\delta(x-a) = \int dx\, F(x+a) f(x+a)\,\delta(x) = F(a) f(a) = \int dx\, F(x) f(a)\,\delta(a-x) \qquad \forall F$$

(iii) $$\int dx\, F(x)\,\delta(\varphi(x)) = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} dx\, F(x)\,\delta(\varphi(x)) = \left[\begin{array}{l} \varphi(x)=y,\ x=\varphi^{-1}(y) \\ dy = \varphi'(x)\,dx \end{array}\right.$$

$$= \sum_i \int_{f(x_i-\varepsilon)}^{f(x_i+\varepsilon)} dy\, \frac{F(x=\varphi^{-1}(y))}{|\varphi'(x=\varphi^{-1}(y))|}\,\delta(y) = \sum_i \frac{F(x_i)}{|\varphi'(x_i)|} = \sum_i \int dx\, F(x)\,\frac{\delta(x-x_i)}{|\varphi'(x_i)|} \qquad \forall F$$

example 2: (1) $$\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right]$$