1.4.2. The space of rank-2 tensors

a) Prove the theorem of ch.1 §4.3: Let \( V \) be a vector space \( V \) of dimension \( n \) over \( K \). Then the space of rank-2 tensors, defined via bilinear forms \( f : V \times V \to K \), forms a vector space of dimension \( n^2 \).

b) Consider the space of bilinear forms \( f \) on \( V \) that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

*hint:* On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

1.4.3. Cross product of 3-vectors

Let \( x, y \in \mathbb{R}^3 \) be vectors, and let \( \epsilon_{ijk} \) be the Levi-Civita symbol. Show that the (covariant) components of the cross product \( x \times y \) are given by

\[
(x \times y)_i = \epsilon_{ijk}x^jy^k
\]

(1 point)

1.4.5. \( \mathbb{R} \) as a metric space

Consider the reals \( \mathbb{R} \) with \( \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( \rho(x, y) = |x - y| \). Show that this definition makes \( \mathbb{R} \) a metric space.

(3 points)

1.4.6. Limits of sequences

a) Show that a sequence in a metric space has at most one limit.

*hint:* Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequency with a limit is a Cauchy sequence.

(3 points)