1.4.9. **Lorentz transformations in** $M_2$

Consider the 2-dimensional Minkowski space $M_2$ with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $2 \times 2$ matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves $g$ invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix $J$ (called the generator of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine $J$ explicitly.

(6 points)

1.4.11. **Special Lorentz transformations in** $M_4$

Consider the Minkowski space $M_4$.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R_i^j \end{pmatrix} \equiv R^\mu_\nu$ (rotations)

   where $R_i^j$ is any Euclidian orthogonal transformation.

ii) $D^\mu_\nu = \begin{pmatrix} \cosh \alpha & \sinh \alpha & 0 & 0 \\ \sinh \alpha & \cosh \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu_\nu$ (Lorentz boost along the $x$-direction)

   with $\alpha \in \mathbb{R}$.

iii) $D^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu_\nu$ (parity)

iv) $D^\mu_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu_\nu$ (time reversal)

.../over
b) Let $L$ be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of $L$, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu_\nu = \delta^\mu_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of $L$.

1.5.1. **Transformations of tensor fields**

a) Consider a covariant rank-$n$ tensor field $t_{i_1...i_n}(x)$ and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how $\tilde{t}_{i_1...i_n}(\tilde{x})$ is related to $t_{i_1...i_n}(x)$.

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector $x_i$ (the case $n = 1$), and (ii) the covariant components of the metric tensor $g_{ij}$.

1.5.2. **Curl and divergence**

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

1.5.3. **Tensor products, and tensor traces**

Prove Propositions 1 and 2 from ch. 1 §5.3.