1.1.1 Russell’s Paradox (B. Russell, 1901)

a) Consider the set \( M \) defined as the set of all sets that do not contain themselves as an element: \( M = \{ x; x \notin x \} \). Discuss why this is a problematic definition.

b) A less abstract version of Russell’s paradox is known as the barber’s paradox: Consider a town where all men either shave themselves, or let the barber shave them and don’t shave themselves. Now consider the statement

\[ \text{The barber is a man in town who shaves all men who do not shave themselves, and only those.} \]

Discuss why this definition of the barber is problematic (assuming there actually is a barber in town).

hint: Ask “Does the barber shave himself?”

c) Suppose the definition of the barber is modified to read

\[ \text{The barber shaves all men in town who do not shave themselves, and only those.} \]

Discuss what this modification does to the paradox.

(3 points)

1.1.2 Distributive property of the union and intersection relations

Show graphically that the relations \( \cup \) and \( \cap \) defined in ch.1, §1.1, def. 3 obey the following distributive properties: For any three sets \( A, B, C \),

\[
A \cap (B \cup C) = (A \cap B) \cup (A \cap C)
\]

\[
A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
\]

(2 points)

1.1.3 Mappings

Are the following \( f : X \rightarrow Y \) true mappings? If so, are they surjective, or injective, or both?

a) \( X = Y = \mathbb{Z} \), \( f(m) = m^2 + 1 \).

b) \( X = Y = \mathbb{N} \), \( f(n) = n + 1 \).

c) \( X = \mathbb{Z}, Y = \mathbb{R} \), \( f(x) = \log x \).

d) \( X = Y = \mathbb{R} \), \( f(x) = e^x \).

(2 points)

1.1.4 Parabolic Mapping

Consider \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) defined by \( f(n) = an^2 + bn + c \), with \( a, b, c \in \mathbb{Z} \).

a) For which triplets \((a, b, c)\) is \( f \) surjective?

b) For which \((a, b, c)\) is \( f \) injective?

(4 points)
1.1.1) a) Suppose it contains itself as a client. Then, by its definition, it does not.
Suppose it does not contain itself as a client. Then, again, by its definition, it does.
There is no third possibility, so the definition is logically self-contradictory.

b) Suppose the barber shaves himself. Then he is shorn by the barber and by definition does not shave himself.
Suppose the barber does not shave himself. Then he is shorn by the barber and hence he does shave himself.
This is the same logical problem as in part a).

c) By dropping the requirement that the barber "is a man in town," the problem goes away: the barber can be a woman, or a man from a different town.

NB: A logically possible conclusion from a) is also that the town has no barber.
1.1.2) \( A \cap (A \cup C) = (A \cap B) \cup (A \cap C) \)
1.1.2 (e) \( f \) is a mapping. It is neither injective (\( f(n) = f(n+1) \)) nor surjective (\( f(n) = f(n+1) \not\in \mathbb{Z} \)).

(b) \( f \) is a mapping. It is not surjective (\( 1 \in \mathbb{N} \) has no pre-image). It is injective, i.e., \( n+1 \) is monotonic.

(c) \( f \) is not a mapping, i.e., \( x \leq 0 \) have no image.

(d) \( f \) is a mapping, i.e., \( e^x \) is defined \( \forall x \in \mathbb{R} \). It is not injective, i.e., \( f(x) > 0 \ \forall x \in \mathbb{R} \). It is injective, i.e., \( e^x \) is monotonic.
1.1.4) a) \( f(n) \) has a critical minimum if \( c + 0 \)
\( \Rightarrow 0 = 0 \) is necessary for \( f \) to be injective.
Now consider \( f(n) = bn + c \)
\( \Rightarrow b \neq 0 \), then \( f(n) = c \) ad has not injective.
\( \Rightarrow b = 2 \) or \( b = -2 \), then \( f(n) \) must equals \( c + 1 \), ed
\( \Rightarrow b = 1 \) for any \( c \), then \( f(n) \) covers all of \( \mathbb{Z} \).
\( \Rightarrow f \) is injective for \( (0, 3, c) \times (0, 1, c \in \mathbb{Z}) \)

2) For \( f \) to be injective, \( f(n) = f(m) \) must imply \( n = m \).
\( \Rightarrow n = m + x \), with \( x \in \mathbb{Z} \).
The \( f(n) = f(m) \) terms the form
\[ cn^2 + bn = cm^2 + bm + c \]
\[ \Rightarrow c(x^2 + (2m + 3)x = 0 \] (x)
\( x = 0 \) is always a solution, while implies \( n = m \).
For \( x \neq 0 \), the only solution of \( (x) \) is
\[ x = -2m - 6 \]
As long as this solution is \( \in \mathbb{Z} \), \( f \) is injective.
\( \Rightarrow \) For \( f \) to be injective, \( b \) must not be divisible by \( c \in \mathbb{Z} \).
\( \Rightarrow f \) is injective for \( (c \in \mathbb{Z}, b \in \mathbb{Z} \setminus 2a, c \in \mathbb{Z}) \)