1.2.3 The group $S_3$

a) Compile the group table for the symmetric group $S_3$. Is $S_3$ abelian?

b) Find all subgroups of $S_3$. Which of these are abelian? (6 points)

1.2.4 Subgroups

Let $(G, \lor)$ be a group and let $H \subset G$ with $H \neq \emptyset$. Show that $H$ is a subgroup of $G$ if and only if $a, b \in H$ implies $a \lor b^{-1} \in H$. (5 points)

1.3.1 Fields

a) Show that the set of rational numbers $\mathbb{Q}$ forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set $F$ with two elements, $F = \{\theta, e\}$. On $F$, define an operation “plus” (+), about which we assume nothing but the defining properties

$$\theta + \theta = \theta , \quad \theta + e = e + \theta = e , \quad e + e = \theta$$

Further, define a second operation “times” (\cdot), about which we assume nothing but the defining properties

$$\theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta , \quad e \cdot e = e$$

Show that with these definitions (and no additional assumptions), $F$ is a field. (7 points)

1.4.1 Function space

Consider the set $C$ of continuous functions $f : [0, 1] \to \mathbb{R}$. Show that by suitably defining an addition on $C$, and a multiplication with real numbers, one can make $C$ an additive vector space over $\mathbb{R}$. (2 points)
The subgroup \( S_3 \) on

\[
P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}
\]

\[
P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}
\]

Will this representation, the Swap table be:

<table>
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<tr>
<th></th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
<th>( P_6 )</th>
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</tbody>
</table>

\( S_3 \) is not abelian: E.g., \( P_3 \circ P_2 = P_5 \), \( P_2 \circ P_3 = P_4 \).
b) Write the jump table from problem 9c. Now write which of $P_2$ not written

5. Case: $\{P_3, P_5, P_4, P_5, P_6\}$ does not write $P_5 = P$
   $\{P_3, P_5, P_4, P_5, P_6\}$ is not closed, i.e. $P_3 \circ P_5 = P_2$
   sum for the other 4 possibilities

4. Case: The rest not write $P_5 \Rightarrow$ We can find
   $\{P_3, P_2, P_3, P_4\}$ not closed, i.e. $P_3 \circ P_2 = P_5$
   $\{P_3, P_2, P_3, P_5\}$ is not closed, i.e. $P_3 \circ P_2 = P_5$
   $\{P_4, P_3, P_4, P_5\}$ is not closed, i.e. $P_4 \circ P_3 = P_5$
   $\{P_5, P_5, P_4, P_5\}$ is not closed, i.e. $P_5 \circ P_5 = P_5$
   sum for the other 6 possibilities

1. Case: Now write $\{P_2, P_4, P_5\}$, which has a jump table

<table>
<thead>
<tr>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
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</thead>
<tbody>
<tr>
<td>$P_2$</td>
<td>$P_4$</td>
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<td>$P_4$</td>
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<td>$P_5$</td>
<td>$P_5$</td>
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This is an obvious misjump!

When $\ldots$

$\{P_3, P_5, P_3\}$ is not closed, i.e. $P_3 \circ P_5 = P_5$

and the same for the other 8 possibilities.
I claim: \( \{P_1, P_2\} \) is an abelian subalgebra.

\( \{P_3, P_4\} \) is not abelian.

\( \{P_5, P_6\} \) is an abelian subalgebra.

\( \{P_7\} \) trivially is an abelian subalgebra.

The subalgebras of \( N_2 \) are:

\[
\begin{align*}
S_2 &= \{(123), (132), (231)\} \\
S_3 &= \{(123), (132)\} \\
S_4 &= \{(123), (123)\} \\
S_5 &= \{(123), (231)\} \\
S_6 &= \{(123), (132)\}
\end{align*}
\]

Any of the above.
1.2.4

(1) Show that \((e, \cdot, H \Rightarrow e \cdot b^{-1} \cdot e \cdot H) \Rightarrow H \leq e \cdot b \cdot e \cdot H\)

\[ e \cdot b \cdot e \cdot H \Rightarrow e \cdot b^{-1} \cdot e \cdot H \]

In particular, \(b \cdot e \cdot H \Rightarrow e \cdot b \cdot e \cdot H \Rightarrow e \cdot b \cdot e \cdot H \Rightarrow b^{-1} \cdot e \cdot H \]

and if \(e \cdot e \) then \(e \cdot b^{-1} = b^{-1} \cdot e \cdot H \)

\(\Rightarrow\) Axim (iii), (iv) from 2.1 on fulfilled

Axim (ii) is fulfilled via \(G \leq H \leq \) then the associativity.

Now we have \(e \cdot b \cdot e \cdot H \Rightarrow e \cdot b^{-1} \cdot e \cdot H \Rightarrow b^{-1} \cdot e \cdot H \) if \(b \cdot e \cdot H \)

\(\Rightarrow\) Axim (i) is fulfilled

\(\Rightarrow\) \(H \leq e \cdot b \cdot e \cdot H \Rightarrow \) The modulle is subset

(2) Show that \((e, \cdot, H \Rightarrow e \cdot b \cdot e \cdot H \) does not imply \(e \cdot b^{-1} \cdot e \cdot H \) \Rightarrow \(H \) is not a group

\[ e \cdot b \cdot e \cdot H \Rightarrow e \cdot b^{-1} \cdot e \cdot H \]

In order for \(H \) to be a group, \(b \cdot e \cdot H \Rightarrow e \cdot b^{-1} \cdot e \cdot H \)

Now we have \(e \cdot b \cdot e \cdot H \), but \(e \cdot b^{-1} \cdot e \cdot H \)

\(\Rightarrow\) Axim (i) is violated \(\Rightarrow\) \(H \) is not a group

\(\Rightarrow\) The modulle is not a group.
1.3.5.1) 

\(Q\) is a group under addition with neutral element \(0 \in Q\):

(i) \(q_1 + q_2 \in Q \neq q_3, q_4 \in Q\)

(ii) Addition is associative and commutative.

(iii) The number \(0\) is a neutral of \(Q\), i.e. \(0 + q = q\) for all \(q \in Q\).

(iv) Let \(q \in Q\). Then \(-q : q + (-q) = 0\).

\(Q\) is also a group under multiplication:

(i) \(q_1 q_2 \in Q \neq q_3, q_4 \in Q\)

(ii) Multiplication is associative and commutative.

(iii) The number \(1\) is a neutral of \(Q\), i.e. \(1 \cdot q = q\) for all \(q \in Q\).

(iv) Let \(q \in Q\). Then \(\frac{1}{q} : q \cdot \frac{1}{q} = 1\).

Finally, ordinary addition and multiplication in \(Q\) are distributive.

\(\therefore Q\) is a commutative field.
b) We need to show that $F$ is a group under addition.

(i) $0, e \in F$ by definition

(ii) $(0+e) + e = e + 0 = e = 0 + (e+e)$

(iii) $+$ is associative

(iv) $e$ is the neutral element by definition

(v) $+$ is commutative by definition

$\Rightarrow F$ is a abelian group under $+$.

We also need to show that $F \setminus \{0\}$ is a group under $\cdot$,

$\Rightarrow F \setminus \{0\}$ is a group under $\cdot$.

Finally, we must check the distributive laws. Let $a, b, c \in F$.

(i) $a(b + c) = ab + ac$ by definition

(ii) $(a + b)c = ac + bc$ by definition

$\Rightarrow F$ is a field.
1.4.1) On $C$, define $(f+g)(x) := f(x) + g(x)$

If $f$ and $g$ are continuous, then so is the sum and defined $(f+g)$

Furthermore, since $f(x) \in \mathbb{R}$, $C$ inherits all of the other group properties from $(\mathbb{R}, +)$

$C$ is an additive group.

Now define multiplication: will scales $\lambda \in \mathbb{R}$ by $(\lambda f)(x) := \lambda f(x)$

If $f$ is continuous, then so is the product defined $(\lambda f)$.

Furthermore, since $\lambda \in \mathbb{R}$ and $f(x) \in \mathbb{R}$, this multiplication will scales is bilinear and associative, as it inherits these properties from $\mathbb{R}$ with ordinary addition and multiplication of numbers.

Finally, $(1f)(x) = 1f(x) = f(x) \forall x \in [0,1] \Rightarrow 1f = f$

$C$ is a $\mathbb{R}$-vector space.