Chapter 10

Tensor Calculus

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1. Scalars, Vectors, Tensors

Certain quantities in physics seem to have absolute significance while other quantities can be defined only *relative to a certain frame of reference.* Mass, density, temperature, and specific heat are represented by *pure numbers,* assigned to certain physical categories. Such quantities are called *scalars.* Other quantities, however, involve the dimensions of space. In a one-dimensional world all measurements would be reducible to scalars, but in a two- or higher-dimensional manifold quantities occur which cannot be measured by pure numbers. They involve magnitude and direction and require a definite frame of reference for analytical characterization, e.g., a vector may be visualized as an arrow put in space. The invariant description of such directed quantities gave rise to a relatively recently developed branch of mathematical physics, called *absolute calculus* or *tensor calculus.* In it vectors are a special case of a more general class of directed quantities, called tensors, which play a fundamental role in the functional relations of the physical universe.

2. Analytic Operations with Vectors

A vector may be visualized as an arrow which has magnitude and direction. In vector analysis, Chap. 9, such a quantity is represented by an algebraic symbol with suitable properties. Certain geometrical operations on two vectors are denoted by $\mathbf{A} + \mathbf{B}$, or $\mathbf{A} - \mathbf{B}$, or $\mathbf{A} \cdot \mathbf{B}$, or $\mathbf{A} \times \mathbf{B}$. The tools of ordinary algebra are thus put into the service of directed quantities. Differentiation and integration are also applicable to certain operations with vectors.

In this procedure some of the basic postulates of ordinary algebra have to be sacrificed. Vector algebra is less simple than ordinary algebra by requiring two kinds of multiplications: the *scalar* product $\mathbf{A} \cdot \mathbf{B}$ and the *vector* product $\mathbf{A} \times \mathbf{B}$. This complication can be avoided by the use of *Hamilton's quaternions* which combine the two kinds of multiplications into one single operation: the product $\mathbf{AB}$ of the two quaternions $\mathbf{A}$ and $\mathbf{B}$. Even so the commutative law of ordinary multiplication $\mathbf{AB} = \mathbf{BA}$ has to be abandoned, although the other postulates of algebra are retained.

3. Unit Vectors; Components

A different and more far-reaching approach is obtained by introducing a system of mutually perpendicular unit vectors for the analysis of vectors. In space of three dimensions three such vectors are necessary and sufficient for the description of an arbitrary vector $\mathbf{A}$. For analytical purposes the three-dimensional nature of physical space is of accidental significance and can be replaced by the concept of an *n-dimensional* space in which n mutually perpendicular unit vectors of the length 1:

$$\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n \tag{10.1}$$

are sufficient for the representation of an arbitrary vector $\mathbf{A}$. The vector $\mathbf{A}$ is now obtained as a linear superposition of the vectors (10.1):

$$\mathbf{A} = a_1\mathbf{U}_1 + a_2\mathbf{U}_2 + \cdots + a_n\mathbf{U}_n \tag{10.2}$$

The quantities

$$a_1, a_2, \ldots, a_n \tag{10.3}$$

called the *components* of the vector $\mathbf{A}$ are obtained by projecting $\mathbf{A}$ on the unit vectors:

$$a_i = \mathbf{A} \cdot \mathbf{U}_i \tag{10.4}$$

While these a_i are ordinary real numbers which satisfy all the postulates of ordinary algebra, they cannot be conceived as an aggregate of scalars since they have additional significance because of their association with the frame of axes (10.1) to which they belong. These components are comparable to the digits of the decimal number 3425. The given digits characterize this number only if the base 10 is given to which they belong. The same digits, if associated with the base 8, belong to an entirely different number; on the other hand, the same number appears in the new form 6541 if the base 8 is adopted. The number remained the same but its components have changed in the new reference system.

Thus a vector is defined by a set of n real numbers (10.3) in relation to a particular frame of axes. An important property of these numbers is the rule of transformation to find the components on changing to some other frame of n unit vectors

$$\bar{\mathbf{U}}_1, \bar{\mathbf{U}}_2, \ldots, \bar{\mathbf{U}}_n \tag{10.5}$$

These rules are developed in the *absolute calculus* or *tensor calculus*, which falls into two main chapters: the algebraic operations with tensors, *tensor algebra*, and the infinitesimal operations with tensor fields, *tensor analysis*.

4. Adjoint Set of Axes

In a more general reference frame the basic vectors need not be mutually perpendicular or of length 1, even though for practical purposes we usually prefer such a system. Tensor calculus uses an arbitrary skew-angular set of basic vectors,

$$\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n \tag{10.6}$$

not restricted in length and mutual positions, except that they be *linearly independent*, i.e., the volume included by them shall not be zero.

Operation with such a system of basic vectors is greatly facilitated by associating with it a second set of basic vectors, called the *adjoint set*. For this new set of vectors the same notation $\mathbf{V}$ is used with the subscripts in an upper position

$$\mathbf{V}^1, \mathbf{V}^2, \ldots, \mathbf{V}^n \tag{10.7}$$

An orthogonal set of axes is characterized by the customary orthogonality conditions:

$$\mathbf{U}_i \cdot \mathbf{U}_k = 0 \qquad i \neq k \tag{10.8}$$

while the normalization of the length of axes to 1 adds the further condition

$$\mathbf{U}_i^2 = 1 \tag{10.9}$$

Although the general set of vectors (10.6) satisfies neither of these two conditions, we can always define a new set of vectors (10.7) by the conditions

$$\mathbf{V}_i \cdot \mathbf{V}^k = 0 \qquad i \neq k$$
$$\text{and} \qquad \mathbf{V}_i \cdot \mathbf{V}^i = 1 \tag{10.10}$$

To any given $\mathbf{V}_i$ these equations are solvable and the solution is unique, provided that the given $\mathbf{V}_i$ are linearly independent.

The original and the adjoint set of vectors are in a dual relation to each other: the adjoint of the adjoint set leads back to the original set. The conditions (10.10) express the mutual orthogonality or *biorthogonality* of the two vector sets $\mathbf{V}_i$ and $\mathbf{V}^i$ and their mutual normalization.

The special advantage of the orthogonal and normalized set of unit vectors $\mathbf{U}_i$ can now be seen in the fact that here the adjoint set $\mathbf{V}^i$ coincides with the original set $\mathbf{V}_i$. Hence an orthogonal and normalized (*orthonormal*) set of unit vectors is *self-adjoint*, thus avoiding the doubling of the fundamental set of vectors.

The adjoint set $\mathbf{V}^i$ can be generated as a linear superposition of the given vectors $\mathbf{V}_i$:

$$\mathbf{V}^i = g^{i1}\mathbf{V}_1 + g^{i2}\mathbf{V}_2 + \cdots + g^{in}\mathbf{V}_n = \sum_{\alpha=1}^{n} g^{i\alpha}\mathbf{V}_\alpha \tag{10.11}$$

Since the $\mathbf{V}_i$ are given, we have the following dot products:

$$\mathbf{V}_i \cdot \mathbf{V}_k = \mathbf{V}_k \cdot \mathbf{V}_i = g_{ik} \tag{10.12}$$

These $g_{ik} = g_{ki}$ form the elements of a symmetric matrix. The conditions (10.10) now demand:

$$\sum_{\alpha=1}^{n} g^{i\alpha} g_{\alpha k} = \delta_{ik} \tag{10.13}$$

(The *Kronecker symbol* δ_{ik} is defined as follows: its value is 1 for $i = k$, and 0 for $i \neq k$.) The matrix of the g^{ik} is the *reciprocal* of the g_{ik} matrix. The existence of the reciprocal matrix demands that the determinant

$$g = \|g_{ik}\| \tag{10.14}$$

shall not be zero. The geometrical significance of this determinant is the square of the volume included by the n base vectors $\mathbf{V}_i$. Since the $\mathbf{V}_i$ are linearly independent, according to our basic assumption, this volume cannot vanish, and the existence (and uniqueness) of the $g^{ik} = g^{ki}$ is guaranteed.

The duality of the adjoint sets permits us to complete (10.11) by the analogous dual equation

$$\mathbf{V}_i = \sum_{\alpha=1}^{n} g_{i\alpha}\mathbf{V}^\alpha \tag{10.15}$$

$$\text{with} \qquad \mathbf{V}^i \cdot \mathbf{V}^k = \mathbf{V}^k \cdot \mathbf{V}^i = g^{ik} \tag{10.16}$$

The symmetric matrices g_{ik} and g^{ik} are fundamental for the general theory of tensors and for Einstein's theory of general relativity.

In the special case of an orthonormal set of axes (10.8) and (10.9) the g_{ik} are reduced to the elements of the unit matrix:

$$g_{ik} = g^{ik} = \delta_{ik} \tag{10.17}$$

and we obtain $\mathbf{V}^i = \mathbf{V}_i$

5. Covariant and Contravariant Components of a Vector

In view of the complete duality of the vectors $\mathbf{V}_i$ and $\mathbf{V}^i$, each set can equally be used for the analysis of a given vector $\mathbf{A}$. We can put

$$\mathbf{A} = a^1\mathbf{V}_1 + a^2\mathbf{V}_2 + \cdots + a^n\mathbf{V}_n = \sum_{\alpha=1}^{n} a^\alpha \mathbf{V}_\alpha \tag{10.18}$$

$$\text{with} \qquad a^i = \mathbf{A} \cdot \mathbf{V}^i \tag{10.19}$$

and likewise

$$\mathbf{A} = a_1\mathbf{V}^1 + a_2\mathbf{V}^2 + \cdots + a_n\mathbf{V}^n = \sum_{\alpha=1}^{n} a_\alpha \mathbf{V}^\alpha \tag{10.20}$$

$$\text{with} \qquad a_i = \mathbf{A} \cdot \mathbf{V}_i \tag{10.21}$$

The a^i and the a_i are two independent sets of components, associated with the same vector $\mathbf{A}$, but expressing that vector in the reference system of the $\mathbf{V}_i$ and in the adjoint reference system of the $\mathbf{V}^i$. The a^i are called the *contravariant*, the a_i the *covariant* components of the same vector $\mathbf{A}$. The relation between these two sets of components can be found with the help of (10.11) and (10.15).

$$a^i = \sum_{\alpha=1}^{n} g^{i\alpha} a_\alpha \qquad (10.22)$$

$$a_i = \sum_{\alpha=1}^{n} g_{i\alpha} a^\alpha \qquad (10.23)$$

If the axes are self-adjoint (orthonormal), the g_{ik} assume the normal values δ_{ik}, and a^i and a_i become identical.

6. Transformation of the Basic Vectors V_i

A different set of basic vectors $\bar{V}_i$ can be expressed in the reference systems of the original V_i, giving rise to relations of the form:

$$\bar{V}_i = \sum_{\alpha=1}^{n} \beta_i^\alpha V_\alpha \qquad (10.24)$$

while the inverse transformation takes the form:

$$V_i = \sum_{\alpha=1}^{n} \bar{\beta}_i^\alpha \bar{V}_\alpha \qquad (10.25)$$

The matrices β_i^k and $\bar{\beta}_i^k$ are reciprocal to each other:

$$\sum_{\alpha=1}^{n} \beta_i^\alpha \bar{\beta}_\alpha^i = \sum_{\alpha=1}^{n} \beta_\alpha^i \bar{\beta}_k^\alpha = \delta_{ik} \qquad (10.26)$$

Existence of the inverse matrix $\bar{\beta}_i^k$ is guaranteed by the demand that the vectors $\bar{V}_i$ shall also be linearly independent.

The definition of the adjoint set of vectors gives

$$\beta_i^k = \bar{V}_i \cdot V^k$$
$$\bar{\beta}_i^k = V_i \cdot \bar{V}^k \qquad (10.27)$$

and the transformation of the adjoint vectors is given by the following equations, dual to (10.24) and (10.25):

$$\bar{V}^i = \sum_{\alpha=1}^{n} \bar{\beta}_\alpha^i V^\alpha$$

$$V^i = \sum_{\alpha=1}^{n} \beta_\alpha^i \bar{V}^\alpha \qquad (10.28)$$

7. Transformation of Vector Components

The vector A can be analyzed in the new set of axes, obtaining

$$\bar{a}_i = A \cdot \bar{V}_i = \sum_{\alpha=1}^{n} \beta_i^\alpha a_\alpha$$

$$\bar{a}^i = A \cdot \bar{V}^i = \sum_{\alpha=1}^{n} \bar{\beta}_\alpha^i a^\alpha \qquad (10.29)$$

The covariant components follow the transformation law of the V_i (*are covariant with* the V_i), while the contravariant components follow the transformation law of the V^i (are *contravariant to* the V_i).

8. Radius Vector R

The position of an ordinary point P in space can be characterized by a set of contravariant coordinates

$$x^1, x^2, \ldots, x^n \qquad (10.30)$$

defined as the contravariant components of the radius vector R:

$$R = x^1 V_1 + x^2 V_2 + \cdots + x^n V_n = \sum_{\alpha=1}^{n} x^\alpha V_\alpha \qquad (10.31)$$

The same point P can likewise be characterized in terms of the covariant coordinates

$$x_1, x_2, \ldots, x_n \qquad (10.32)$$

defined by the covariant components of the radius vector R:

$$R = x_1 V^1 + x_2 V^2 + \cdots + x_n V^n = \sum_{\alpha=1}^{n} x_\alpha V^\alpha \qquad (10.33)$$

The square of the radius vector R has an important geometrical significance. It expresses the square of the distance of the point P from the origin in terms of the coordinates of P. Making use of the definition of the g_{ik} and g^{ik} according to (10.12) and (10.16):

$$R^2 = \sum_{i=1}^{n} \sum_{k=1}^{n} g_{ik} x^i x^k = \sum_{i,k=1}^{n} g_{ik} x^i x^k \qquad (10.34)$$

Similarly

$$R^2 = \sum_{i,k=1}^{n} g^{ik} x_i x_k \qquad (10.35)$$

An expression of the form (10.34) is called a *quadratic form* of the variables x^i. The particular quadratic form which defines the square of the distance of the point x^i from the origin is called the *fundamental metrical form*.

The relation between the x_i and the x^i is established on the basis of (10.22) and (10.23):

$$x^i = \sum_{\alpha=1}^{n} g^{i\alpha} x_\alpha \qquad (10.36)$$

$$x_i = \sum_{\alpha=1}^{n} g_{i\alpha} x^\alpha \qquad (10.37)$$

9. Abstract Definition of a Vector

A more abstract definition of a vector may be given which brings the central principle of tensor calculus, *the principle of invariance*, into sharp focus:

We start with the variables $x^1, x^2, \ldots, x^n$, which characterize the position of an arbitrary point P in space, and assume that the square of the distance s of that point from the origin of the reference system is given by the quadratic form

$$s^2 = \sum_{i,k=1}^{n} g_{ik} x^i x^k \qquad (10.38)$$

We then introduce the covariant x_i by the definition

$$x_i = \sum_{\alpha=1}^{n} g_{i\alpha} x^\alpha \qquad (10.39)$$

Hence

$$s^2 = \sum_{\alpha=1}^{n} x_\alpha x^\alpha \qquad (10.40)$$

We now consider the *linear form* of the variables x^i

$$A = a_1 x^1 + \cdots + a_n x^n = \sum_{\alpha=1}^{n} a_\alpha x^\alpha \qquad (10.41)$$

and define the coefficients $a_1, \ldots, a_n$ of this linear form as the covariant components of a vector **A**. The same form A can also be written in terms of the x_i:

$$A = a^1 x_1 + \cdots + a^n x_n = \sum_{\alpha=1}^{n} a^\alpha x_\alpha \qquad (10.42)$$

with

$$a^i = \sum_{\alpha=1}^{n} g^{i\alpha} a_\alpha \qquad (10.43)$$

thus defining the *contravariant* components of the same vector **A**.

If the vector **A** is regarded as constant force, then the physical significance of the scalar A is the *work* done by the force during the displacement OP = **R**. In the abstract definition of a vector the justification of calling **A** a vector is taken from the fact that the work of the force **A** for arbitrary positions of the radius vector **R** appears as a linear form of the coordinates x^i. The coefficients of this form define the covariant components of the force **A**.

A mere set of numbers $a_i, \ldots, a_n$ does not establish a vector since these coefficients have significance only in connection with a given set of coordinate axes. The abstract definition of a vector takes this property of the vector components into account since the linear form A is established solely in connection with the variables $x^1, x^2, \ldots, x^n$. In particular we consider first purely *rectilinear* systems, i.e., coordinate systems whose parameter lines are parallel straight lines. This means in terms of the x^i that we consider arbitrary *linear* transformations of the variables x^i:

$$\bar{x}^i = \sum_{\alpha=1}^{n} \bar{\beta}^i_\alpha x^\alpha \qquad (10.44)$$

with non-vanishing determinant $\|\bar{\beta}^i_\alpha\|$. The inverse transformation is then given by

$$x^i = \sum_{\alpha=1}^{n} \beta^i_\alpha \bar{x}^\alpha \qquad (10.45)$$

where the matrix β^i_k is the reciprocal of the matrix $\bar{\beta}^i_k$.

The transformation of the covariant x_i is established by the principle that the *bilinear form*

$$s^2 = \sum_{\alpha=1}^{n} x_\alpha x^\alpha \qquad (10.46)$$

shall be an invariant of the transformation:

$$\sum_{\alpha=1}^{n} x_\alpha x^\alpha = \sum_{\alpha=1}^{n} \bar{x}_\alpha \bar{x}^\alpha \qquad (10.47)$$

This principle establishes the transformation of the x_i as the reciprocal of the transformation of the x^i:

$$\bar{x}_i = \sum_{\alpha=1}^{n} \beta^\alpha_i x_\alpha \qquad (10.48)$$

$$x_i = \sum_{\alpha=1}^{n} \bar{\beta}^\alpha_i \bar{x}_\alpha \qquad (10.49)$$

Transformation of vector components is established by the principle that the linear form A shall be an invariant of the transformation:

$$\sum_{\alpha=1}^{n} a_\alpha x^\alpha = \sum_{\alpha=1}^{n} \bar{a}_\alpha \bar{x}^\alpha \qquad (10.50)$$

The individual coefficients a_i change their values if the frame of axes is changed. The value of the *entire linear form* A, however, must *not* be influenced by the transformation, no matter what the position of the point P is. This principle establishes the transformation law of the a_i in the form

$$\bar{a}_i = \sum_{\alpha=1}^{n} \beta^\alpha_i a_\alpha \qquad (10.51)$$

The transformation of the contravariant a^i is similarly established by the invariance of the linear form (10.41):

$$\bar{a}^i = \sum_{\alpha=1}^{n} \bar{\beta}^i_\alpha a^\alpha \qquad (10.52)$$

The duality of the components a_i and a^i and their transformation laws *without any reference to unit vectors* have been developed by using the following tools: (1) The definition of a vector on the basis of an invariant linear form. (2) The existence of a distance square defined by an invariant form of second order.

10. Invariants and Covariants

In theory of relativity the distinction between quantities which change with the reference system (*covariants*) and quantities which do not change with the reference system (*invariants*) is of paramount importance. The coefficients of a linear form are covariants since they depend on the reference system employed. The *entire linear form*, however, is an *invariant* of the transformation which does *not* change its value in any rectilinear reference system and thus has absolute significance. In the prerelativistic phase of physics certain quantities which belong to the realm of covariants were treated as invariants, and vice versa. In particular, the time t was considered as an absolute, unchangeable variable which does not participate in any transformations, while in fact nature forms a four-dimensional manifold of space *and* time. This relegates the time t into the realm of a fourth coordinate which is transformed together with the three space variables x,y,z. The orthogonal transformations of 3-space, characterized by the invariance of the quadratic form $x^2 + y^2 + z^2$, were enlarged to the orthogonal transformations of 4-space, characterized by the invariance of the quadratic form

$$x^2 + y^2 + z^2 - c^2t^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - c^2\bar{t}^2 \quad (10.53)$$

where c is the velocity of light which plays the role of a scale factor of the fourth dimension and becomes thus one of the basic constants of nature. The physical significance of the invariance principle (10.53) is that light travels in *any* nonaccelerated reference system with the same velocity c, in every direction, irrespective of the motion of the observer.

In general relativity the expression (10.53), which introduces a Euclidean geometry into the four-dimensional space-time world, is recognized as a macroscopic approximation to reality since the actual metric of the universe is of the Riemannian type and has to be developed on the basis of general tensor calculus.

11. Abstract Definition of a Tensor

Invariant algebraic forms of first order are only a special example of the much wider class of invariant algebraic forms of *any* order. This gives a natural introduction of the tensor concept: a tensor of mth order is defined with the help of an *invariant algebraic form of order m*. The coefficients of this form define the components of the tensor, covariant if the variables are the contravariant coordinates x^i, and contravariant if the variables are the covariant coordinates x_i. A vector is thus a special case appearing as a tensor of first order.

12. Tensors of Second Order

Tensors of second order occur particularly often in the mathematical description of natural phenomena. For example, the elastic stress tensor is a symmetric tensor of second order, as well as the Maxwellian electric stress tensor. In the theory of relativity the electromagnetic field strength is an antisymmetric tensor of second order in space-time.

All these tensors represent mathematically the coefficients of an invariant algebraic form of second order:

$$A = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}x^\alpha x^\beta \quad (10.54)$$

This quadratic form has only $n(n+1)/2$ independent elements, since the terms $a_{ik}x^ix^k$ and $a_{ki}x^kx^i$ combine into one. We make the tensor unique by adding the symmetry condition

$$a_{ki} = a_{ik} \quad (10.55)$$

and speak of a *symmetric tensor*.

A general tensor of second order is defined by the following device: consider two *different* positions of the radius vector **R**, say x^i and y^i, and define the form

$$A = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}x^\alpha y^\beta \quad (10.56)$$

The terms with a_{ik} and a_{ki} are now independent and the form defines n^2 separate elements.

The symmetry pattern (10.55) of the coefficients can be augmented by the pattern

$$a_{ki} = -a_{ik} \quad (10.57)$$

which defines an antisymmetric tensor of second order having $n(n-1)/2$ independent components. The six independent components of such a tensor in 4-space combine the electric and magnetic field strength into one entity.

The same tensor of second order may be given in covariant or contravariant or mixed form, according to the nature of the variables employed:

$$\begin{aligned} A &= \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}x^\alpha y^\beta \\ &= \sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}x_\alpha y_\beta \\ &= \sum_{\alpha,\beta=1}^{n} a^\alpha{}_\beta x_\alpha y^\beta \\ &= \sum_{\alpha,\beta=1}^{n} a_\alpha{}^\beta x^\alpha y_\beta \end{aligned} \quad (10.58)$$

13. Einstein Sum Convention

The homogeneous notation of the variables and the consistent use of lower and upper indices for the distinction of covariant and contravariant components and variables contributed greatly to the systematic development of tensor calculus. An additional operational simplification was introduced by Einstein. In all previous formulas the position of the indices is such that the summation occurs over an index which in one factor is in the upper position and in the other factor in the lower position. Now whenever the same index appears twice in opposite positions in a formula, we shall *automatically sum over that index.* Hence the notation

$$a_\alpha x^\alpha$$

shall mean

$$\sum_{\alpha=1}^{n} a_\alpha x^\alpha$$

The same convention holds if a product contains more than one pair of equal indices, e.g., the double sum

$$\sum_{i,k=1}^{n} a_{ik}x^i y^k$$

is now written in the form

$$a_{ik}x^i y^k$$

This simplification greatly facilitates the symbolic manipulations with tensors.

14. Tensor Algebra

Algebraic operations with tensors are an immediate consequence of the general definition of a tensor as the coefficients of an invariant algebraic form of the order m. The general definition makes use of m independent positions of the radius vector **R**:

$$x^i,\ y^k,\ \ldots\ ,\ z^m$$

Moreover, any of these sets of variables may be put in covariant form

$$x_i,\ y_k,\ \ldots\ ,\ z_m$$

Every set of variables is associated with one subscript or superscript in the coefficients, corresponding indices being always in juxtaposition, for example,

$$A = a_{ik\cdots m}x^i y^k \cdots z^m$$
$$A = a^i_k{}^{\cdots m} x_i y^k \cdots z_m$$

Addition of Tensors. The sum of two invariants is again an invariant. The sum of the two forms

$$a_i x^i + b_i x^i = (a_i + b_i)x^i \tag{10.59}$$

defines a new invariant form of first order. The quantities

$$c_i = a_i + b_i \tag{10.60}$$

form the covariant components of a new tensor of first order.

Generally two tensors of the same order whose components are in homologous positions can be added; the sum defines a new tensor of the same order, with the same distribution of covariant and contravariant components:

$$a_i{}^{km} + b_i{}^{km} = c_i{}^{km} \tag{10.61}$$

Multiplication by a Constant. The multiplication of all tensor components by the same constant defines a new tensor of the same order and same distribution of indices:

$$\alpha a_i{}^{km} = c_i{}^{km} \tag{10.62}$$

Multiplication of Two Tensors. The product of two invariant algebraic forms gives once more an invariant algebraic form. The order of the new form is the sum of the orders of the composing factors:

$$(a_i x^i)(b_k y^k) = a_i b_k x^i y^k \tag{10.63}$$

Hence

$$a_i b_k = c_{ik} \tag{10.64}$$

defines a tensor of second order, covariant in both indices. Generally the product of any two tensors, with any distribution of covariant and contravariant components, yields a tensor whose order is the sum of the order of the composing factors and whose indices exactly repeat the entire set of composing indices, e.g., the product

$$a_i{}^m b_k{}^p = c_i{}^m{}_k{}^p \tag{10.65}$$

yields a tensor of fourth order, covariant in i,k, contravariant in m,p.

Transposition of Indices. If in the definition of an invariant algebraic form the positions of the radius vector are exchanged, we once more obtain an invariant algebraic form of the same order. For example, if

$$A = a_{ik}x^i y^k \tag{10.66}$$

is an invariant,

$$B = a_{ik}y^i x^k = a_{ki}x^i y^k \tag{10.67}$$

is also an invariant. This shows that if a_{ik} is a covariant tensor of second order,

$$b_{ik} = a_{ki} \tag{10.68}$$

is also a covariant tensor of second order. Generally it is permissible to exchange any two indices which are both in the upper or both in the lower position.

With the help of this operation we can always decompose a covariant or a contravariant tensor of even order into the sum of two tensors; the one symmetric, the other antisymmetric in one pair of indices. For example, the covariant tensor of second order a_{ik} may be written in the form:

$$a_{ik} = \tfrac{1}{2}(a_{ik} + a_{ki}) + \tfrac{1}{2}(a_{ik} - a_{ki}) \tag{10.69}$$

The first tensor on the right side is symmetric [see (10.55)], the second antisymmetric [see (10.57)] in i, k.

By the same operation two covariant vectors a_i and b_k give rise to a symmetric tensor of second order

$$c_{ik} = a_i b_k + a_k b_i \tag{10.70}$$

and an antisymmetric tensor of second order

$$d_{ik} = a_i b_k - a_k b_i \tag{10.71}$$

Raising and Lowering Indices. The general definition of a tensor of mth order involved m independent positions of the radius vector R, which could be given in either covariant or contravariant form. However, the general relations (10.36) and (10.37) between covariant and contravariant coordinates make it possible to change any covariant index to a contravariant index, and vice versa. This involves a homologous change in the position of the corresponding index of the associated coefficient.

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The raising of a covariant index occurs by the process

$$a^i \cdots = g^{i\alpha} a_\alpha \cdots \tag{10.72}$$

while the *lowering* of a contravariant index occurs by the process

$$a_i \cdots = g_{i\alpha} a^\alpha \cdots \tag{10.73}$$

The dots indicate any combination of covariant or contravariant indices which do not participate in the operation and which are carried along without any change.

Contraction of a Tensor. The general definition of a tensor as an invariant algebraic form of a certain order includes the transformation law of the components if the variables are subjected to an arbitrary nonsingular linear transformation. All contravariant variables x^i, y^i, . . . follow the same transformation matrix β^i_α, while all covariant variables $x_i, y_i \cdots$ follow the same transformation matrix $\bar{\beta}^\alpha_i$ which is the reciprocal of β^i_α; [see (10.45) and (10.49)]. The transformation law of the tensor components $a_{ik}{}^m \cdots$ is identical with the transformation law of the product

$$x_i y_k z^m \cdots \tag{10.74}$$

Now perform the following operation: Equate one covariant and one contravariant index of a tensor and perform the summation over this index. We might choose, for example, the indices k and m in the example (10.74) and form the quantities

$$a_{i\alpha}{}^\alpha \cdots \tag{10.75}$$

which follow the transformation law of the product

$$x_i(y_\alpha z^\alpha) \cdots \tag{10.76}$$

The bilinear form

$$y_\alpha z^\alpha \tag{10.77}$$

is an *invariant* of a linear transformation. Hence the factor in parentheses in (10.76) behaves like a *constant* during the transformation, and therefore the transformation of the quantities (10.75) follows the transformation law of the product (10.73) omitting the indices k and m. This is equivalent to the statement that the quantities (10.75) form the components of a tensor which has the same indices as the original tensor but omitting the two indices k and m. Thus the operation

$$b_i \cdots = a_{i\alpha}{}^\alpha \cdots \tag{10.78}$$

called *contraction*, generates a new tensor whose order is *lowered by 2* compared with the original tensor.

In the case of a tensor of second order, contraction results in a tensor of zeroth order, giving a scalar or invariant:

$$a = a^\alpha{}_\alpha \tag{10.79}$$

If in particular the tensor $a^i{}_k$ is defined as the product of the two vectors b^i and c_k, we obtain the invariant

$$a = bc \tag{10.80}$$

This invariant is the scalar or dot product $\mathbf{B} \cdot \mathbf{C}$ of the two vectors **B** and **C**.

15. Determinant Tensor

Consider n independent positions of the radius vector R and form the product of the following two determinants, composing rows by rows:

$$\begin{vmatrix} x^1 & x^2 & \cdots & x^n \\ y^1 & y^2 & \cdots & y^n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z^1 & z^2 & \cdots & z^n \end{vmatrix} \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z_1 & z_2 & \cdots & z_n \end{vmatrix} \tag{10.81}$$

The product is a determinant whose elements are invariants, hence it is an invariant. Moreover:

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z_1 & z_2 & \cdots & z_n \end{vmatrix} = \begin{vmatrix} x^1 & x^2 & \cdots & x^n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z^1 & z^2 & \cdots & z^n \end{vmatrix} \cdot \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix} \tag{10.82}$$

Substituting (10.82) in (10.81) and taking the square root, we obtain

$$\sqrt{g} \begin{vmatrix} x^1 & x^2 & \cdots & x^n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z^1 & z^2 & \cdots & z^n \end{vmatrix} = \text{invariant} \tag{10.83}$$

Similarly

$$\frac{1}{\sqrt{g}} \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z_1 & z_2 & \cdots & z_n \end{vmatrix} = \text{invariant} \tag{10.84}$$

From the theory, a determinant of the form (10.83) or (10.84) may be written as an algebraic form of nth order:

$$\begin{vmatrix} x^1 & x^2 & \cdots & x^n \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ z^1 & z^2 & \cdots & z^n \end{vmatrix} = \epsilon_{ik\cdots m} x^i y^k \cdots z^m \tag{10.85}$$

where $\epsilon_{ik\cdots m}$ vanishes for any combination of indices which are not all different from each other, while the nonvanishing $\epsilon_{ik\cdots m}$ are defined as $+1$ if $ik \cdots m$ represents an *even* permutation of the numbers 1,2, . . . ,n, and -1 if $ik \cdots m$ represents an *odd* permutation of the numbers 1,2, . . . ,n.

Hence in any n-dimensional manifold there exists a tensor of nth order, antisymmetric in any pair of indices which has the covariant components

$$\delta_{ik\cdots m} = \sqrt{g}\, \epsilon_{ik\cdots m} \tag{10.86}$$

or the contravariant components

$$\delta^{ik\cdots m} = \frac{1}{\sqrt{g}} \epsilon_{ik\cdots m} \tag{10.87}$$

This tensor is called the *determinant tensor* or *permutation tensor*.

16. Dual Tensor

Multiply a given tensor of mth order, covariant in all its indices, by the contravariant tensor (10.87), contracting over all the indices of the given tensor. The result is a completely contravariant tensor of order $n - m$, antisymmetric in any pair of indices:

$$a^{*pq\cdots s} = a_{\alpha\beta\cdots\gamma}\delta^{\alpha\beta\cdots\gamma pq\cdots s} \tag{10.88}$$

where a^* is called the *dual* of the tensor a. A similar construction applies to the completely covariant tensor if the original tensor is completely contravariant.

Of particular interest is the application of this operation to 3-space and to 4-space. In 3-space the antisymmetric tensor

$$c_{ik} = \tfrac{1}{2}(a_i b_k - a_k b_i) \tag{10.89}$$

is associated with the vectors **A** and **B**. The dual of this tensor becomes a tensor of the order $3 - 2 = 1$, that is, a vector. The contravariant components of this vector are

$$\begin{aligned} c^{*1} &= \frac{1}{\sqrt{g}}(a_2 b_3 - a_3 b_2) \\ c^{*2} &= \frac{1}{\sqrt{g}}(a_3 b_1 - a_1 b_3) \\ c^{*3} &= \frac{1}{\sqrt{g}}(a_1 b_2 - a_2 b_1) \end{aligned} \tag{10.90}$$

which is the customary cross product $\mathbf{A} \times \mathbf{B}$ of vector algebra. This method of associating a third vector to two given vectors is restricted to 3-space because the cross product of two vectors is basically an antisymmetric tensor of second order associated with two vectors according to (10.89). In 3-space the dual of this tensor is a vector, giving rise to the vector components (10.90).

In 4-space the dual of an antisymmetric tensor of second order is again an antisymmetric tensor of second order. This relation is fundamental for the relativistic interpretation of the duality of the Maxwell electromagnetic equations (see Sec. 24).

17. Tensor Fields

The linear algebraic form (10.41) could be interpreted as the work of the force **A** during the displacement $\mathbf{R} = OP$. This required that **A** be a constant force. If a *field* of force is given which changes its magnitude and direction continuously from point to point, we have to think of the infinitesimal displacement x^i which remains in the neighborhood of the point P, the displacement being taken between the points $P = x^i$ and $P' = x^i + dx^i$. The work of the force **A** is then given by the *differential form*

$$A = a_\alpha\, dx^\alpha \tag{10.91}$$

The coefficients of this differential form are no longer constants but continuous functions of the coordinates $x^1, x^2, \ldots, x^n$.

By changing from algebraic forms to differential forms it is possible to extend the realm of tensor operations from constant tensors to tensor fields. Everything remains valid as before with the understanding that all operations of tensor algebra are now performed *at a definite point* x^i of the field. The differentials dx^i can be interpreted as local coordinates of the point P', measured from the center P. The infinitesimal displacement from P to P' eliminates the variable character of the field, since for such displacements the tensor field assumes the behavior of a constant tensor.

For present discussions the field concept will not be extended to the coordinates x^i themselves. These will still be assumed to be *rectilinear coordinates* which extend to the entire space. Hence the transformation from the x^i to the $\bar{x}^i$ is still a *linear* transformation, and the transformation matrix of the differentials dx^i is the same as the transformation matrix of the coordinates x^i themselves.

The transition from algebraic to differential forms does not modify any of the previous results. The only difference is that components of vectors and tensors are now functions of the point P.

The field concept does *not* extend, however, to the metrical tensor g_{ik}. Since the coordinates are rectilinear, the expression (10.34) for the finite distance $S = OP$ is still valid. The differential form of this equation:

$$(dR)^2 = ds^2 = g_{ik}\, dx^i\, dx^k \tag{10.92}$$

defines the square of the line element ds. This *line element* ds is associated with the two neighboring points $P = x^i$ and $P' = x^i + dx^i$ and defines the infinitesimal distance between these two points. The g_{ik} coefficients of this quadratic differential form are constants throughout the field.

18. Differentiation of a Tensor

The abstract definition of a tensor of mth order involves m independent positions $dy^i, dz^k \cdots du^p$ of the *infinitesimal* radius vector $d\mathbf{R}$. The definition occurs with the help of the invariant differential form

$$A = a_{ik\cdots p}\, dy^i\, dz^k \cdots du^p \tag{10.93}$$

where the coefficients $a_{ik\cdots p}$ are functions of the coordinates $x^1, x^2, \ldots, x^n$.

Since the differential of an invariant is again an invariant, we can form the infinitesimal change of A between two neighboring points P and P' of the field. This gives the new invariant

$$dA = \frac{\partial a_{ik\cdots p}}{\partial x^\alpha}\, dy^i\, dz^k \cdots du^p\, dx^\alpha \tag{10.94}$$

which is a differential form of the order $m + 1$. Hence by definition we have obtained a new tensor of the order $m + 1$:

$$b_{ik\cdots pq} = \frac{\partial a_{ik\cdots p}}{\partial x^q} \tag{10.95}$$

In order to indicate that this new tensor, called the *covariant derivative* of the tensor $a_{ik\cdots p}$, originated from that tensor, we do not use a new letter for its designation but adopt the following method of notation:

$$a_{ik\cdots p,q} = \frac{\partial a_{ik\cdots p}}{\partial x^q} \qquad (10.96)$$

The same procedure holds if some or all the indices of the given tensor are contravariant. The differentiation of a tensor is the *only typical operation* of tensor analysis. All operations of tensor analysis are a combination of the differentiation of a tensor discussed in this paragraph, and the previous algebraic operations, discussed in Sec. 14.

19. Covariant Derivative of the Metrical Tensor

The metrical tensor g_{ik} forms a symmetric tensor of second order. Since in a rectilinear reference system the g_{ik} are *constants*, the covariant derivative of the tensor g_{ik} *vanishes* at every point of the n-dimensional manifold:

$$g_{ik,m} = \frac{\partial g_{ik}}{\partial x^m} = 0 \qquad (10.97)$$

20. Principles of Special and General Relativity

Einstein formulated the principle of special relativity which required that all reference systems in uniform motion relative to each other shall be equivalent for the formulation of the laws of nature. This requires that the equations of mathematical physics shall have invariance with respect to an arbitrary linear transformation of the four variables x,y,z,t.

In 1916 Einstein formulated the principle of general relativity (based on the equivalence of heavy mass and inertial mass) which required that arbitrary reference systems in arbitrary motion relative to each other shall be equivalent for the formulation of the laws of nature. This requires that the equations of mathematical physics shall have invariance with respect to ***arbitrary curvilinear transformations*** of the four variables x,y,z,t.

The tools of tensor calculus were in harmony with the principle of general relativity. These tools are in intimate relation to the concepts of Riemannian geometry and brought the importance of the geometry into sharp focus. Einstein applied the mathematical investigations of Riemann to the physical universe and discovered the theory of general relativity which explained mass, energy, and gravity in purely geometrical terms and gave theoretical physics a fundamentally new turn.

21. Curvilinear Transformations

In place of rectilinear coordinates x^i a more general class of curvilinear coordinates $\bar{x}^i$ will now be used characterized by an arbitrary *point transformation*

$$\bar{x}^i = \bar{f}^i(x^1,x^2, \ldots ,x^n) \qquad (10.98)$$

where the $\bar{f}^i(x^1, \ldots ,x^n)$ are given as arbitrary continuous and twice differentiable functions of the old variables x^i, with nonvanishing Jacobian. The inverse of the transformation (10.98) takes the form

$$x^i = f^i(\bar{x}^1,\bar{x}^2, \ldots ,\bar{x}^n) \qquad (10.99)$$

The relation between the *differentials* of the variables remains *linear:*

$$d\bar{x}^i = \frac{\partial \bar{f}^i}{\partial x^\alpha}\, dx^\alpha \qquad (10.100)$$

This suffices for tensor analysis since invariance properties may be established from differential forms.

The new feature associated with the use of curvilinear coordinates is the fact that the matrix of the transformation

$$\bar{\beta}^i_k = \frac{\partial \bar{f}^i}{\partial x^k} \qquad (10.101)$$

now changes from point to point and hence is a field quantity. The geometrical significance of this change is that the local reference systems, characterized by the dx^i, are no longer in parallel orientation to each other. This, however, is irrevelant for the operations of tensor algebra which are restricted to one definite point of the manifold. The only operation which becomes essentially modified by the introduction of curvilinear coordinates is the differentiation of a tensor, since this operation involves relations between field quantities at neighboring points of the manifold.

22. Covariant Derivative of a Tensor

Consider an arbitrary position dy^i of the infinitesimal radius vector dR. The corresponding differentials $d\bar{y}^i$ in the curvilinear system become

$$d\bar{y}^i = \bar{\beta}^i_\alpha\, dy^\alpha \qquad (10.102)$$

Considering the dy^α as constants, the $d\bar{y}^i$ are *not* constants since the factor $\bar{\beta}^i_\alpha$ changes from point to point. This gives:

$$d^2\bar{y}^i = \frac{\partial \bar{\beta}^i_\alpha}{\partial \bar{x}^\nu}\, dy^\alpha\, d\bar{x}^\nu = \beta^\alpha_\mu \frac{\partial \bar{\beta}^i_\alpha}{\partial \bar{x}^\nu}\, d\bar{y}^\mu\, d\bar{x}^\nu$$
$$= -\bar{\beta}^i_\alpha \frac{\partial \beta^\alpha_\mu}{\partial \bar{x}^\nu}\, d\bar{y}^\mu\, d\bar{x}^\nu \qquad (10.103)$$

But

$$\frac{\partial \beta^\alpha_\mu}{\partial \bar{x}^\nu} = \frac{\partial^2 f^\alpha}{\partial \bar{x}^\mu\, \partial \bar{x}^\nu} \qquad (10.104)$$

and hence, introducing the auxiliary quantities

$$\Gamma^m_{ik} = \bar{\beta}^m_\alpha \frac{\partial \beta^\alpha_i}{\partial \bar{x}^k} = \beta^m_\alpha \frac{\partial^2 f}{\partial \bar{x}^i\, \partial \bar{x}^k} \qquad (10.105)$$

we notice that these quantities (which do *not* form a tensor of third order, in spite of the analogous notation) are symmetric in i,k:

$$\Gamma^m_{ik} = \Gamma^m_{ki} \qquad (10.106)$$

With the help of these quantities the relation (10.103) becomes

$$d^2\bar{y}^i = -\Gamma^i_{\alpha\beta}\, d\bar{y}^\alpha\, d\bar{x}^\beta \qquad (10.107)$$

The corresponding transformation law of the covariant differentials

$$d\bar{y}_i = \beta^\alpha_i\, dy_\alpha \qquad (10.108)$$

yields

$$d^2\bar{y}_i = \Gamma^\alpha_{i\beta}\, d\bar{y}_\alpha\, d\bar{x}^\beta \qquad (10.109)$$

We now consider the invariant differential form

$$A = \bar{a}_\alpha \, d\bar{y}^\alpha \tag{10.110}$$

written down in an arbitrary curvilinear system. From the differential of this invariant, we derive the covariant derivative of the vector a_i. Now we have to differentiate the second factor too, replacing $d^2\bar{y}^i$ by (10.107). Thus the covariant derivative of the vector a_i in an arbitrary curvilinear system is

$$a_{i,k} = \frac{\partial a_i}{\partial x^k} - \Gamma^\alpha_{ik} a_\alpha \tag{10.111}$$

Similarly the invariant form $a^\alpha \, dy$ yields:

$$a^i{}_{,k} = \frac{\partial a^i}{\partial x^k} + \Gamma^i_{\alpha k} a^\alpha \tag{10.112}$$

Generally, applying the same principle to a differential form of arbitrary order, we obtain the result that *every index of the tensor gives rise to a correction term*. If the index is covariant, the correction term follows the pattern of Eq. (10.111), if contravariant, the pattern of Eq. (10.112). The remaining indices are carried along unchanged, e.g.,

$$a^i{}_{k,m} = \frac{\partial a^i{}_k}{\partial x^m} + \Gamma^i_{\alpha m} a^\alpha{}_k - \Gamma^\alpha_{km} a^i{}_\alpha \tag{10.113}$$

23. Covariant Derivative of the Metrical Tensor

If curvilinear coordinates are introduced, the g_{ik} become field quantities. The transformation of Eq. (10.92) to curvilinear coordinates gives

$$\bar{g}_{ik} = g_{\mu\nu}\beta^\mu_i\beta^\nu_k = g_{\mu\nu}\frac{\partial f^\mu}{\partial \bar{x}^i}\frac{\partial f^\nu}{\partial \bar{x}^k} \tag{10.114}$$

which reveals the field character of the new g_{ik}. Nevertheless, the covariant derivative of g_{ik} must vanish since a tensor which is zero in any rectilinear reference system remains zero in *every* reference system. This gives the important relation

$$\frac{\partial g_{ik}}{\partial x^m} - \Gamma^\alpha_{ik} g_{\alpha k} - \Gamma^\alpha_{km} g_{i\alpha} = 0 \tag{10.115}$$

We introduce the so-called *Christoffel symbols of the first kind:*

$$\Gamma^\alpha_{im} g_{\alpha k} = \left[{ik \atop m}\right] \tag{10.116}$$

and rewrite (10.115) with these symbols. We also know the symmetry of the Christoffel symbols in the two upper indices, in view of (10.106). We thus deduce by a simple algebraic manipulation:

$$\left[{ik \atop m}\right] = \frac{1}{2}\left(\frac{\partial g_{im}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}\right) \tag{10.117}$$

and obtain the important result that the *auxiliary quantities* Γ^m_{ik}, originally defined in terms of the transformation equations to curvilinear coordinates (cf. (10.105)), *are expressible in terms of the metric associated with that curvilinear system.*

We can thus completely abandon the transformation equations which originally gave rise to a curvilinear system. If we possess the metrical tensor g_{ik} associated with that reference system, we can immediately form the quantities

$$\Gamma^m_{ik} = \left[{ik \atop \alpha}\right] g^{\alpha m} \tag{10.118}$$

and thus obtain all the tools for the formation of covariant derivatives. The entire edifice of absolute calculus for arbitrary curvilinear coordinates can thus be erected on the basis of the invariance of differential forms, plus the existence of the metrical tensor g_{ik}.

24. Fundamental Differential Invariants and Covariants of Mathematical Physics

Apart from the fundamental importance of general relativity, the study of absolute calculus has also a purely practical value. The differential equations of mathematical physics have to be solved frequently under boundary conditions which demand the introduction of the proper kind of curvilinear coordinates, such as spherical, cylindrical, parabolic, or other coordinates. The tools of absolute calculus put us in the position to write down the basic differential equations of physics in *any* reference system.

The differential operators of absolute calculus are complicated by the appearance of Γ-quantities. However, in many of the fundamental differential operators of mathematical physics these quantities enter in a highly simplified manner. We list below the most important differential invariants and covariants of mathematical physics. In deducing these expressions, the following relation is of great usefulness:

$$\Gamma^\alpha_{\alpha i} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^i} g^{\mu\nu} = \frac{1}{2g}\frac{\partial g}{\partial x^i} = \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^i} \tag{10.119}$$

where g is defined by (10.14).

Divergence of a Vector. The following scalar can be associated with a vector field, called the *divergence* of the vector **A**:

$$a^\alpha{}_{,\alpha} = \operatorname{div} a = \frac{\partial \sqrt{g}\, a^\alpha}{\sqrt{g}\, \partial x^\alpha} \tag{10.120}$$

Laplacian Operator $\Delta\phi$: Let the vector a_i of (10.120) be the gradient of the scalar function ϕ:

$$a_i = \frac{\partial \phi}{\partial x^i} \tag{10.121}$$

The divergence of this vector gives the invariant Laplacian operator

$$\Delta\phi = \operatorname{div}\operatorname{grad}\phi = \frac{\partial \sqrt{g}\, g^{\alpha\beta}(\partial\phi/\partial x^\alpha)}{\sqrt{g}\, \partial x^\beta} \tag{10.122}$$

Divergence of a Symmetric Tensor T^{ik}. Let $T^{ik} = T^{ki}$ be a symmetric tensor of second order. We form the following vector, called the divergence of the tensor T^{ik}:

$$T^{i\alpha}{}_{,\alpha} = \frac{\partial \sqrt{g}\, T^{i\alpha}}{\sqrt{g}\, \partial x^\alpha} - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^i} T^{\mu\nu} \tag{10.123}$$

This differential covariant is of fundamental importance since the conservation law of momentum and energy appears in field physics in the form that the divergence of the symmetric matter tensor T^{ik} vanishes.

Divergence of an Antisymmetric Tensor F^{ik}. Let $F^{ik} = -F^{ki}$ be an antisymmetric tensor of second order (e.g., electromagnetic field strength). Then

$$F^{i\alpha}{}_{,\alpha} = \frac{\partial \sqrt{g}\, F^{i\alpha}}{\sqrt{g}\, \partial x^\alpha} \tag{10.124}$$

25. Maxwell Electromagnetic Equations

The Maxwellian equations of the electromagnetic field, considered relativistically, split into two vector equations. The first equation states that the divergence of the electromagnetic field strength $F^{ik} = -F^{ki}$ is equal to the current density vector:

$$\frac{\partial \sqrt{g}\, F^{i\alpha}}{\sqrt{g}\, \partial x^\alpha} = S^i \tag{10.125}$$

The second equation states that the divergence of the dual tensor vanishes:

$$\frac{\partial \sqrt{g}\, F^{*i\alpha}}{\partial x^\alpha} = 0 \tag{10.126}$$

Considering the definition (10.88) of the dual tensor, we obtain

$$\frac{\partial F_{\mu\nu}}{\partial x^\alpha}\, \epsilon_{\mu\nu\alpha i} = 0$$

The remarkable feature of this equation is that it does not contain any metrical quantity explicitly. It is solvable by putting

$$F_{ik} = \frac{\partial \phi_i}{\partial x^k} - \frac{\partial \phi_k}{\partial x^i} \tag{10.127}$$

which is likewise free of metrical quantities. ϕ_i is called the *vector potential*. It is subject to the conservation law of electricity which has the consequence that the divergence of ϕ_i vanishes:

$$\frac{\partial \sqrt{g}\, \phi_\alpha g^{\alpha\beta}}{\partial x^\beta} = 0 \tag{10.128}$$

26. Curvature Tensor of Riemann

We consider the second covariant derivative of a vector a, and differentiate once in the sequence jk and once in the sequence kj. If this operation is performed in a rectilinear reference system, we find that the result is in both cases the same. Hence we would expect that also in a curvilinear system the two tensors $a_{i,jk}$ and $a_{i,kj}$ will agree. In fact the result comes out as follows:

$$a_{i,jk} - a_{i,kj} = -R^\alpha{}_{ijk} a_\alpha \tag{10.129}$$

$$\text{where } R^\alpha{}_{ijk} = \frac{\partial \Gamma^\alpha_{ij}}{\partial x^k} - \frac{\partial \Gamma^\alpha_{ik}}{\partial x^j} + \Gamma^\mu_{ij}\Gamma^\alpha_{k\mu} - \Gamma^\mu_{ik}\Gamma^\alpha_{j\mu} \tag{10.130}$$

Since the left side of (10.129) is a tensor of third order, covariant in i,j,k, the factor of a_α must be a tensor of fourth order, contravariant in α and covariant in i,j,k. It is completely composed of the Γ-quantities and thus is of a completely *metrical* character.

The apparent paradox that the tensor (10.129) vanishes in a rectilinear system but does not vanish in a curvilinear system is caused by the fact that the assumption of a universally rectilinear reference system strongly prejudices the metrical character of a manifold. The metrical tensor of a manifold *may* have the form (10.114), in which case it is derived from an originally *constant* g_{ik}. In this case we have a metrical geometry which satisfies the postulates of Euclidean geometry. But it is also possible that the g_{ik} of a curvilinear reference system are prescribed as *some* field quantities, without demanding that they shall be of specific form (10.114). Riemann in 1854 established the far-reaching idea of a metrical manifold which is characterized by a quadratic differential form *without* any further restrictions except for the natural conditions of continuity and differentiability. A geometry of this kind, called *Riemannian geometry*, is Euclidean only in *infinitesimal* portions of space but not in finite portions. The Euclidean type of geometry is a specially simple example of Riemannian geometry in which the tensor (10.130), called the *Riemann-Christoffel curvature tensor* or briefly the *Riemann tensor*, vanishes identically. The second covariant derivative of a tensor is then independent of the sequence of differentiations, which is not true in the more general metrical pattern of Riemannian geometry.

27. Properties of Riemann Tensor

The curvature tensor R_{ijkm} is characterized by the following algebraic symmetry properties.

It is antisymmetric in the first pair of indices:

$$R_{jikm} = -R_{ijkm} \tag{10.131}$$

It is antisymmetric in the second pair of indices:

$$R_{ijmk} = -R_{ijkm} \tag{10.132}$$

It is symmetric with respect to an exchange of the first and second pair of indices:

$$R_{kmij} = R_{ijkm} \tag{10.133}$$

It satisfies the *cyclic identity*

$$R_{ijkm} + R_{ikmj} + R_{imjk} = 0 \tag{10.134}$$

These symmetry properties reduce the number of algebraically independent components to $n^2(n^2 - 1)/12$. Hence the number of independent components is 1 in 2-space, 6 in 3-space, and 20 in 4-space.

Differential properties of the curvature tensor. R_{ijkm} satisfies the following differential identity, called the *Bianchi identity:*

$$R_{ijkm,n} + R_{ijmn,k} + R_{ijnk,m} = 0 \tag{10.135}$$

28. Contracted Curvature Tensor

Einstein thought that the metrical tensor g_{ik} should be considered as a field quantity and subjected to field equations. These field equations must take the form of some partial differential equations which have invariant significance in all curvilinear reference systems. The curvature tensor of Riemann did not seem suitable for this purpose since it is a tensor of *fourth* order, with $n^2(n^2 - 1)/12$ algebraically independent components, while the metrical tensor is only a symmetric tensor of *second* order, with $n(n+1)/2$ independent components. Any statement in terms of the full Riemann tensor would thus be strongly overdetermined.

A contraction changes a tensor of fourth order to a tensor of second order. The contraction over the first two indices of the Riemann tensor gives zero, on account of the antisymmetric nature of these two indices. However, a contraction over the first and third (or second and fourth) indices generates a new tensor of second order, called the *Einstein tensor*, which is symmetric in i and k. We denote this tensor again by the letter R since the possession of only two indices distinguishes this tensor R_{ik} sufficiently from the full Riemann tensor R_{ijkm}:

$$R_{ik} = R_i{}^{\alpha}{}_{k\alpha} = \frac{\partial^2 \log \sqrt{g}}{\partial x^i\, \partial x^k} - \frac{\partial \sqrt{g}\, \Gamma^{\alpha}_{ik}}{\sqrt{g}\, \partial x^{\alpha}} + \Gamma^{\alpha}_{i\beta}\Gamma^{\beta}_{k\alpha} \tag{10.136}$$

A second contraction generates an invariant, called the *scalar Riemannian curvature* or the *Gaussian curvature*. We denote it by the letter R, without any indices:

$$R = R^{\alpha}{}_{\alpha} = \left[\frac{\partial^2 \log \sqrt{g}}{\partial x^{\mu}\, \partial x^{\nu}} - \frac{\partial \sqrt{g}\, \Gamma^{\alpha}_{\mu\nu}}{\sqrt{g}\, \partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right] g^{\mu\nu} \tag{10.137}$$

In two dimensions, where the Riemann tensor has only *one* independent component, the scalar Gaussian curvature R is sufficient for the characterization of a Riemannian manifold.

In three dimensions, where the Riemann tensor has *six* independent components, the contracted curvature tensor R_{ik} is sufficient for a full characterization of a Riemannian manifold.

In four dimensions, where the Riemann tensor has 20 independent components, the contracted tensor R_{ik} with its 10 components does not give a full characterization of a Riemannian manifold. Here the vanishing of R_{ik} does *not* necessitate (as in two and three dimensions) the flattening out of space due to the vanishing of the full Riemann tensor.

29. The Matter Tensor of Einstein

If in the Bianchi identity (10.135) we raise the first two indices and then contract over i,k and also over j,m, we obtain a remarkable result which can be written as follows:

$$T^{i\alpha}{}_{,\alpha} = 0 \tag{10.138}$$

where the symmetric tensor T_{ik} is defined by

$$T_{ik} = R_{ik} - \tfrac{1}{2} R g_{ik} \tag{10.139}$$

The identity (10.138) says that the divergence of the symmetric tensor T_{ik} is zero. But this is exactly the form in which the conservation law of momentum and energy appears if matter is considered as a field quantity which is continuously distributed over space-time. Einstein equated the tensor T_{ik} with the matter tensor of theoretical physics and thus obtained a purely geometrical interpretation of mass, energy and momentum. Mass density or energy density can thus be conceived in terms of the curvature of a four-dimensional Riemannian manifold. The Riemannian curvature is particularly high at such portions of space-time where there is matter while in empty space the tensor T_{ik} vanishes.

30. Einstein's Theory of Gravity

Matter is concentrated in the stars and planets, with interplanetary spaces free of matter. The statement that the matter tensor vanishes yields the field equation

$$R_{ik} = 0 \tag{10.140}$$

for determination of the g_{ik}. While in two or three dimensions Eq. (10.140) would reduce the g_{ik} to constants, in four dimensions this is no longer the case. The field equations (10.140) represent a grand generalization of the Laplacian equation

$$\Delta\phi = 0 \tag{10.141}$$

which characterizes the Newtonian potential. Solution of the field equations (10.140) under spherically symmetric conditions gave the Riemannian metric of space-time generated by the sun. The motion of the planets according to the law of inertia—that means along shortest lines in this Riemannian manifold—gave a complete description of gravitational phenomena, including the fundamental identity of gravitational and inertial mass, without further hypotheses. Moreover, the theory predicted a number of minute effects which were not included in the previous theories: in particular, the red shift of the spectral lines in strong gravitational fields, the deflection of light at the limb of the sun, and a slight secular precession of the planetary ellipses, an effect which in the case of Mercury assumes measurable proportions. All these predictions of the theory have been corroborated by astronomical observations.

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