PHYS 610 Mathematical Methods for Scientists F 2024

Midterm Exam 10/23/2024
due 11/06/2024

This take-home problem serves as the Midterm. It runs concurrently with Problems Sets #4 and #5. Treat it like any homework assignment, but please do not collaborate on this one. Submit your solution via email to dbelitz@uoregon.edu by 5pm on Wednesday, November 6.

I.1.8 Inverse Mappings (15 pts)

Note: This problem proves the statement from ch. I § 1.2 Remark (8). It is strongly recommended that you prove the Lemma first, then the Proposition, and then the Theorem. If you get hopelessly stuck on one of those, assume that it's true, clearly state that fact, and continue.

Prove the following

Theorem: Let X and Y be sets, and let $f : X \to Y$ be a bijective mapping. Then $\exists!\ f^{-1} : Y \to X$ called the *inverse* of f such that

$$f \circ f^{-1} = \mathrm{id}_Y \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{id}_X \ .$$

f^{-1} is also bijective, and its inverse is $(f^{-1})^{-1} = f$.

Hint: It is useful to first prove the following

Proposition: Let $f : X \to Y$ be surjective. Then $\exists\ g : Y \to X$ such that g is injective and $f \circ g = \mathrm{id}_Y$.

For this, in turn, it is useful to first prove the

Lemma: Let $f : X \to Y$, $g : Y \to X$, and $f \circ g = \mathrm{id}_Y$. Then f is surjective and g is injective.

Note: There is a point in the proof of the proposition where you need to make use, consciously or otherwise, of the

Axiom of Choice: Let I be an index set and $X_i \neq \varnothing \quad \forall i \in I$. Then the cartesian product $\prod_{i\in I} X_i \neq \varnothing$. Or, equivalently and in plain English: Given a class of nonempty sets there exists a "choice function" that picks from each set one of its elements. (This sounds obvious, but it is not, and the realization that it isn't has a very interesting history.)

To apply this in the proof of the Proposition, consider Y the index set. Then the axiom of choice guarantees the existence of the mapping g you are looking for.