PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 4 10/30/2024
due 11/06/2024

I.4.7. Banach space

Let B be a K-vector space (K = $\mathbb{R}$ or $\mathbb{C}$) with null vector θ. Let $||\ldots|| \ : \ \mathrm{B} \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0 \ \forall \ x \in \mathrm{B}$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y|| \ \forall \ x, y \in \mathrm{B}$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \ \forall \ x \in \mathrm{B}, \lambda \in \mathrm{K}$.

Then we call $||\ldots||$ a **norm** on B, and $||x||$ the **norm** of x.

Further define a mapping $d \ : \ \mathrm{B} \times \mathrm{B} \to \mathbb{R}$ by

$d(x, y) := ||x - y|| \ \forall \ x, y \in \mathrm{B}$

Then we call $d(x, y)$ the **distance** between x and y.

a) Show that d is a metric in the sense of ch. I §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space B with distance/metric d is complete, then we call B a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 Example (11), and you don't have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let B^* be the dual space of B, i.e., the space of linear forms ℓ on B, and define a norm of ℓ by the "sup norm"

$$||\ell|| := \sup_{||x||=1} \{|\ell(x)|\}$$

c) Show that the such defined norm on B^* is a norm in the sense of the norm defined on B above.

(In case you're wondering: B^* is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)