PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 7 11/13/2024
due 11/20/2024

II.2.1. **Lindhard function**

Consider the function $f : \mathbb{C} \to \mathbb{C}$ (which plays an important role in the theory of many-electron systems) defined by

$$f(z) = \log\left(\frac{z-1}{z+1}\right)$$

The *spectrum* $f'' : \mathbb{R} \to \mathbb{R}$ and the *reactive part* $f' : \mathbb{R} \to \mathbb{R}$ of f are defined by

$$f''(\omega) := \frac{1}{2i}\left[f(\omega + i0) - f(\omega - i0)\right] \qquad , \qquad f'(\omega) := \frac{1}{2}\left[f(\omega + i0) + f(\omega - i0)\right]$$

where $f(\omega \pm i0) := \lim_{\epsilon\to 0} f(\omega \pm i\epsilon)$.

a) Show that f' and f'' are indeed real-valued functions.

b) Determine f'' and f' explicitly, and plot them for $-3 < \omega < 3$.

c) Show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\, \frac{f''(\omega)}{\omega - z} = f(z)$$

(5 points)

II.2.2. **Another causal function**

The function considered in Problem 2.2.1 is an example of a class of complex functions called *causal functions* that are important in the theory of many-particle systems. Another member of this class is

$$g(z) = \sqrt{z^2 - 1} - z$$

Determine the spectrum and the reactive part of $g(z)$, and plot them for $-3 < \omega < 3$.

(3 points)

II.2.3. **Proof of the Cauchy-Riemann Theorem**

Prove the Cauchy-Riemann theorem from ch.2 §2.2:

a) Let $f(z) = f'(z', z'') + if''(z', z'')$ be analytic everywhere in $\Omega \subseteq \mathbb{C}$. Show that the Cauchy-Riemann equations

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \qquad \text{and} \qquad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}$$

hold $\forall z \in \Omega$.

hint: Start with the difference quotient $(f(z) - f(z_0))/(z - z_0)$ and require that it's limit for $z \to z_0$ exists if z_0 is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

b) Let the Cauchy-Riemann equations hold in a point $z_0 \in \Omega$. Show that this implies that f is analytic in the point z_0.

hint: Consider $f(z) - f(z_0)$ and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series about z_0.

(8 points)

II.2.4. **Exponentials**

Consider the exponential function

$$f(z) = e^z = e^{z' + iz''}$$

a) Show that $f(z)$ is analytic everywhere in $\mathbb{C}$.

b) Convince yourself explicitly that the real and imaginary parts of f obey Laplace's differential equation.

c) Show that $df/dz|_z = f(z)$.

d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad , \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

are analytic everywhere in $\mathbb{C}$, and that

$$\frac{d}{dz}\cos z = -\sin z \quad , \quad \frac{d}{dz}\sin z = \cos z \ .$$

(4 points)