PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 8 11/20/2024
due 11/27/2024

II.3.1. Laurent series

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter Theorem 2 in ch. 2 §3.2.

(3 points)

II.3.2. Applications of the residue theorem

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx \, \frac{1}{x^4 + 1}$$

b)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

c)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \frac{1}{1 + x^2}$$

and check your results numerically..

Let $a \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Use the residue theorem to show that

d)

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

e)

$$\int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{x^2 + a^2}$$

and define its Cauchy principal value by

$$\lim_{R \to 0} [\int_{-\infty}^{-R} dx \, f(x) + \int_{R}^{\infty} dx \, f(x)]$$

with $f(x) = 1/x(x^2 + a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \to 0$.

(17 points)

II.3.3. Matsubara frequency sum

Let $f(z)$ have simple poles at z_j ($j = 1, 2, \ldots$), and no other singularities. Let $f(|z| \to \infty)$ go to zero faster then $1/z$. Consider the infinite sum

$$S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$$

with $\Omega_n = 2\pi T n$ and $T > 0$. Show that

$$S = \sum_j n(z_j) \, \mathrm{Res} \, f(z_j)$$

where $n(z) = 1/(e^{z/T} - 1)$ is the Bose distribution function.

hint: Show that $n(z)$ has simple poles at $z = i\Omega_n$, and integrate $n(z)\, f(z)$ over an infinite circle centered on the origin.

note: Sums of this form are important in finite-temperature quantum field theory. In this context, T is the temperature and Ω_n is called a "bosonic Matsubara frequency".

(3 points)