PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 9 11/27/2024
due 12/04/2024

II.4.1. 1-d Fourier transforms

Consider a function f of one real variable x. Calculate the Fourier transforms $\hat{f}(k) = \int dx\, e^{-ikx}\, f(x)$ of the following functions:

a) $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$.

b) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$.

c) $f(x) = e^{-(x/x_0)^2}$.

(3 points)

II.4.2. 3-d Fourier transforms

Consider a function f of one vector variable $\boldsymbol{x} \in I\!R^3$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x}) \quad .$$

Calculate the Fourier transforms of the following functions:

a) $f(\boldsymbol{x}) = \begin{cases} 1 & \text{for } r < r_0 \qquad (r = |\boldsymbol{x}|) \\ 0 & \text{otherwise} \end{cases}$.

b) $f(\boldsymbol{x}) = 1/r$.

hint: Consider $g(\boldsymbol{x}) = \frac{1}{r}\, e^{-r/r_0}$ and let $r_0 \to \infty$.

c) $f(\boldsymbol{x}) = e^{-\alpha \boldsymbol{x}^2}$ with $\alpha \in I\!R$, $\alpha > 0$.

(3 points)

II.4.3. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\; e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm \int dt\, e^{izt}\, f_{\pm}(t) \qquad (\pm \text{ for } \mathrm{sgn}(\mathrm{Im}\, z) = \pm 1)$$

with z a complex frequency and $f_{\pm}(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) = \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon \to 0} \epsilon/(x^2 + \epsilon^2) = \pi\, \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in ch. II §4.5.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

II.4.5. Generalized functions derived from generalized functions

Prove Proposition 7 in ch.II §4.4, which says

Let $f(x)$ and $g(x)$ be generalized functions defined by sequences $f_n(x)$ and $g_n(x)$. Then

a) the sum $f(x) + g(x)$ defined by the sequence $f_n(x) + g_n(x)$, and

b) the derivative $f'(x)$ defined by the sequence $f_n'(x)$, and

c) $h(x) = f(ax + b)$ defined by the sequence $f_n(ax + b)$, and

d) $\varphi(x)\, f(x)$ defined by the sequence $\varphi(x)\, f_n(x)$ with φ a fairly good function,and

e) $\hat{f}(k)$ defined by the sequence $\hat{f}_n(k) = FT(f_n)(k)$.

are all generalized functions.

(7 points)