I.1.1 **Russell's Paradox (B. Russell, 1901)**

a) Consider the set M defined as the set of all sets that do not contain themselves as an element: $M = \{x; x \notin x\}$. Discuss why this is a problematic definition.

b) A less abstract version of Russell's paradox is known as the barber's paradox: Consider a town where all men either shave themselves, or let the barber shave them and don't shave themselves. Now consider the statement

   The barber is a man in town who shaves all men who do not shave themselves, and only those.

   Discuss why this definition of the barber is problematic (assuming there actually is a barber in town).

   hint: Ask "Does the barber shave himself?"

c) Suppose the definition of the barber is modified to read

   The barber shaves all men in town who do not shave themselves, and only those.

   Discuss what this modification does to the paradox.

(3 points)

Solution

a) Suppose M contains itself as an element. Then, by its definition, it does not.
   Suppose M does not contain itself as an element. Then, again by its definition, it does.
   There is no third possibility, so the definition is logically self-contradictory. 1pt

b) Suppose the barber shaves himself. This means that he is shaved by the barber and hence does **not** shave himself.
   Suppose the barber does not shave himself. Then he is shaved by the barber and hence **does** shave himself.
   This is the same logical problem as in part a). 1pt

c) By dropping the requirement that the barber is "a man in town" the problem goes away: The barber may be a woman, or a man from a different town. 1pt
   Note: Another logically possible conclusion is that the town has no barber.

I.1.2 **Distributive property of the union and intersection relations**

a) Prove the distributive properties from ch.I §1.1 Remark 5 viz.,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

for any three sets A, B, C.

b) Illustrate these properties by means of representative Venn diagrams.

(5 points)

Solution

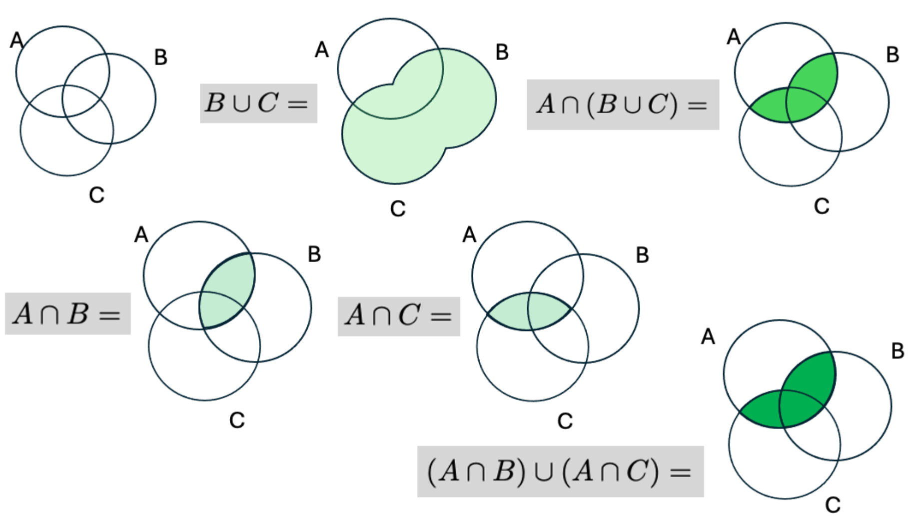
a) Let $x \in A \cap (B \cup C)$. $\Rightarrow x \in A \wedge (x \in B \vee x \in C)$.
1st case: $x \in B \wedge x \in A \Rightarrow x \in A \cap B \Rightarrow x \in (A \cap B) \cup (A \cap C)$
2nd case: $x \in C \wedge x \in A \Rightarrow x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$
$\Rightarrow$ In either case, $x \in (A \cap B) \cup (A \cap C)$
$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ 1pt

Let $x \in (A \cap B) \cup (A \cap C)$. $\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$
1st case: $x \in A \wedge x \in B$. $\Rightarrow x \in A \wedge x \in B \cup C$
2nd case: $x \in A \wedge x \in C$. $\Rightarrow x \in A \wedge x \in B \cup C$
$\Rightarrow$ In either case, $x \in A \cap (B \cup C)$.
$\Rightarrow (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ 1pt

Exactly analogous arguments prove the second set equality. 1pt

b) The first equality can be illustrated as follows: 1pt



The second equality can be illustrated analogously. 1pt

NB: These drawings don't prove anything, they just make the result plausible. In order to graphically prove the statements one would have to consider all possible overlapping situations.

I.1.3 **Mappings**

Are the following $f : X \to Y$ true mappings? If so, are they surjective, or injective, or both?

a) $X = Y = \mathbb{Z}$, $\quad f(m) = m^2 + 1$.

b) $X = Y = \mathbb{N}$, $\quad f(n) = n + 1$.

c) $X = \mathbb{Z}$, $Y = \mathbb{R}$, $\quad f(x) = \log x$.

d) $X = Y = \mathbb{R}$, $\quad f(x) = e^x$.

(2 points)

Solution

a) f is a mapping. It is neither surjective (since $f(n) \geq 1 \; \forall b \in \mathbb{Z}$) nor injective (since $f(-n) = f(n) \; \forall n \in \mathbb{Z}$). 0.5 pt

b) f is a mapping. It is not surjective ($1 \in \mathbb{N}$ has no pre-image). It is injective, since it is monotonic. 0.5 pt

c) f is not a mapping, since $x \leq 0$ has no image. 0.5 pt

d) f is a mapping, since e^x is defined $\forall x \in \mathbb{R}$. It is not surjective, since $f(x) > 0 \, \forall x \in \mathbb{R}$. It is injective, since it is monotonic. 0.5 pt

I.1.4 **Parabolic Mapping**

Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = a\,n^2 + b\,n + c$, with $a, b, c \in \mathbb{Z}$.

a) For which triplets (a, b, c) is f surjective?

b) For which (a, b, c) is f injective?

(4 points)

Solution

a) $f(n)$ has a global extremum if $a \neq 0$. $\Rightarrow$ $a = 0$ is necessary for f to be surjective.
Now consider $f(n) = bn + c$.
If $b = 0$, then $f(n) \equiv c$ and hence f is not surjective.
If $b \geq 2$ or $b \leq -2$, then $f(n)$ never equals $c \pm 1$, and hence f is not surjective.
If $b = \pm 1$ with arbitrary c, then $f(n)$ covers all of $\mathbb{Z}$.

$\Rightarrow$ f is surjective for $(a, b, c) = (0, \pm 1, c \in \mathbb{Z})$. 2pts

b) For f to be injective, $f(n) = f(m)$ must imply $n = m$.
Let $n = m + x$ with $x \in \mathbb{Z}$. Then $f(n) = f(m)$ takes the form

$$am^2 + bm = am^2 + 2axm + ax^2 + bm + bx$$

$$\Rightarrow \quad ax^2 + (2am + b)x = 0 \qquad (*)$$

$x = 0$ is always a solution of (*), which implies $n = m$.
For $x \neq 0$ the only solution of (*) is

$$x = -2m - b/a$$

As long as this solution is $\notin \mathbb{Z}$, f is injective.
$\Rightarrow$ For f to be injective, b must not be divisible by a in $\mathbb{Z}$.

$\Rightarrow$ f is injective for $(a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus a\mathbb{Z},\ c \in \mathbb{Z})$. 2pts