I.4.2. The space of rank-2 tensors

a) Prove the theorem of ch.1 §4.3: Let V be a vector space V of dimension n over K. Then the space of rank-2 tensors, defined via bilinear forms $f : V \times V \to K$, forms a vector space of dimension n^2.

b) Consider the space of bilinear forms f on V that is equivalent to the space of rank-2 tensors, and construct a basis of that space.

hint: On the space of tensors, define a suitable addition and multiplication with scalars, and construct a basis of the resulting vector space.

(5 points)

Solution

a) We know that the rank-2 tensors are one-to-one correspondent to bilinear forms $f(x, y)$. On the set of bilinear forms, define an addition by

$$(f + g)(x, y) := f(x, y) + g(x, y)$$

This makes the set of forms an additive group. Also define a multiplication with scalars $\lambda \in K$ by

$$(\lambda\, f)(x, y) := \lambda\, f(x, y)$$

This makes the set of forms a K-vector space. On the space of rank-2 tensors $t, u, \ldots$ this corresponds to defining the tensor $t + u$ as the tensor with coordinates

$$(t + u)_{ij} = t_{ij} + u_{ij}$$ 1pt

and the tensor $\lambda\, t$ as the one with coordinates

$$(\lambda\, t)_{ij} = \lambda\, t_{ij}$$

The space of tensors is now a K-vector space. 1pt

Consider a basis $\{e_i\{$ of V, and construct n^2 tensors

$$E_{ij}^{kl} = \delta_i^{\ k}\, \delta_j^{\ l}$$

Define a tensor t as a linear combination of the E_{ij},

$$t = \sum_{ij} \tau^{ij}\, E_{ij}$$

with coefficients $\tau^{ij} \in K$. This tensor has coordinates

$$t^{kl} = \sum_{ij} \tau^{ij}\, (E_{ij})^{kl} = \tau^{kl}$$

$\Rightarrow$ Any rank-2 tensor can be written as a linear combination of the E_{ij}, with the coordinates t^{ij} of t as the coefficients:

$$t = \sum_{ij} t^{ij}\, E_{ij}$$

$\Rightarrow$ The E_{ij} span the space. 1pt

Now, in order for t to be the null tensor, all of its coordinates must be zero, so $t = 0$ implies $t^{ij} = 0 \; \forall i, j$.

$\Rightarrow$ The E_{ij} are linearly independent.

$\Rightarrow$ The n^2 rank-2 tensors E_{ij} form a basis of the space of rank-2 tensors, and hence the space has dimension n^2.

1pt

b) Let f_{ij} be the bilinear form that corresponds to the tensor E_{ij}. Then

$$f_{ij}(e_k, e_l) = (E_{ij})_{kl} = \delta_{ij}\,\delta_{kl}$$

with δ_{ij} the Kronecker symbol.

For arbitrary $x, y \in V$ we have

$$f_{ij}(x, y) = x^k y^l f(e_k, e_l) = x^k y^l \delta_{ik}\delta_{jl} = x^i y^j$$

$\Rightarrow$ The set of n^2 bilinear forms f_{ij} defined by

$$f_{ij}(x, y) = x^i y^j$$

forms a basis of the space of bilinear forms. 1pt

I.4.5. $\mathbb{R}$ as a metric space

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

Solution

Positive definiteness and symmetry are obvious. 1pt

Now prove the triangle inequality:

Proof. By definition of $|x|$ we have $xy \leq |x| \cdot |y| \; \forall \, x, y \in \mathbb{R}$. Therefore,

$$0 \leq 2(x-y)(z-y) + 2\,|x-y| \cdot |z-y|$$ 1pt

And hence

$$\begin{aligned}
(x-z)^2 &= x^2 - 2xz + z^2 \\
&\leq x^2 - 2xz + z^2 + 2(x-y)(z-y) + 2|x-y| \cdot |z-y| \\
&= x^2 - 2xz + z^2 + 2(x-y)z - 2(x-y)y + 2|x-y| \cdot |z-y| \\
&= x^2 - 2xz + z^2 + 2xz - 2xy + 2y^2 - 2yz + 2|x-y| \cdot |z-y| \\
&= x^2 - 2xy + y^2 + y^2 - 2yz + z^2 + 2|x-y| \cdot |z-y| \\
&= (x-y)^2 + (y-z)^2 + 2|x-y| \cdot |z-y| \\
&= (|x-y| + |y-z|)^2
\end{aligned}$$

But $(x-z)^2 \geq 0$, and hence we have the triangle inequality

$$|x-z| \leq |x-y| + |y-z|$$

□

1pt

I.4.6. Limits of sequences

a) Show that a sequence in a metric space has at most one limit.

hint: Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequence with a limit is a Cauchy sequence.

(3 points)

Solution

a) Prove uniqueness:

Proof. Let x_n be a sequence. Suppose $x_n \Rightarrow x^*$ and $x_n \Rightarrow y^*$. Then

$$\rho(x^*, y^*) \leq \rho(x^*, x_n) + \rho(y^*, x_n) \quad \forall x_n \quad \text{by the triangle equation}$$

But

$$\lim_{n\to\infty} \rho(x^*, x_n) = \lim_{n\to\infty} \rho(y^*, x_n) = 0$$

and hence

$$\rho(x^*, y^*) = 0 \qquad \Rightarrow \qquad x^* = y^*$$

□

1pt

b) Prove Cauchy-ness:

Proof. Let x_n have a limit x^*: $x_n \Rightarrow x^*$. Then

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*)$$ 1pt

Let $\delta > 0$. Then $\exists N \in \mathbb{N}$ such that $\rho(x_n, x^*) < \delta \quad \forall n > N$.

Now let $\epsilon > 0$ and choose $\delta = \epsilon/2$. Then $\exists N > 0$ such that

$$\rho(x_n, x_m) \leq \rho(x_n, x^*) + \rho(x_m, x^*) < \epsilon/2 + \epsilon/2 = \epsilon$$

provided $n, m > N$. □

1pt