I.4.7. Banach space

Let B be a K-vector space (K = $\mathbb{R}$ or $\mathbb{C}$) with null vector θ. Let $||\ldots|| \ : \ \mathrm{B} \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0 \ \forall \ x \in \mathrm{B}$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y|| \ \forall \ x, y \in \mathrm{B}$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \ \forall \ x \in \mathrm{B}, \lambda \in \mathrm{K}$.

Then we call $||\ldots||$ a **norm** on B, and $||x||$ the **norm** of x.

Further define a mapping $d \ : \ \mathrm{B} \times \mathrm{B} \to \mathbb{R}$ by

$d(x, y) := ||x - y|| \ \forall \ x, y \in \mathrm{B}$

Then we call $d(x, y)$ the **distance** between x and y.

a) Show that d is a metric in the sense of ch. I §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space B with distance/metric d is complete, then we call B a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 Example (11), and you don't have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let B^* be the dual space of B, i.e., the space of linear forms ℓ on B, and define a norm of ℓ by the "sup norm"

$$||\ell|| := \sup_{||x||=1} \{|\ell(x)|\}$$

c) Show that the such defined norm on B^* is a norm in the sense of the norm defined on B above.

(In case you're wondering: B^* is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)

Solution

a) $d(x, y) = \|x - y\| \geq 0 \ \forall x, y \in \mathrm{B}$ by property (i) of the norm.

Also, $d(x, y) = 0$ iff $x - y = \theta \ \Leftrightarrow x = y$.

$\Rightarrow$ positive definiteness ✓

Furthermore,

$$\begin{aligned} d(y, x) = \|y - x\| = \|(-(x - y)\| &= \|x - y\| \qquad \text{by property (iii) of the norm} \\ &= d(x, y) \end{aligned}$$

$\Rightarrow$ symmetry ✓

Finally

$$\begin{aligned} d(y, z) = \|y - z\| = \|x - y + y - z\| &\leq \|x - y\| + \|y - z\| \qquad \text{by property (ii) of the norm} \\ &= d(x, y) + d(y, z) \end{aligned}$$

$\Rightarrow$ triangle inequality ✓ 1pt

b) Consider $\mathbb{R}$ as an $\mathbb{R}$-vector space and define

$$\|x\| := |x| \quad \forall x \in \mathbb{R}$$

Then $\|\ldots\| : \mathbb{R} \to \mathbb{R}$ has all of the properties required of a norm. Furthermore, it follows from §4.5 example (11) that every Cauchy sequence has a limit, so $\mathbb{R}$ is complete and hence a B-space.

The same arguments apply to $\mathbb{C}$ with a norm defined by

$$\|z\| := |z| = \sqrt{z_1^2 + z_2^2}$$

This makes $\mathbb{C}$ a B-space (assuming completeness). 1pt

c) Check the requirements for a norm:

(i) $\|\ell\| = \sup_{\|x\|=1} \{|\ell(x)|\} \quad \Rightarrow \quad \|\ell\| \geq 0 \quad \text{since } |\ell(x)| \geq 0$ 0.5 pt

The null vector in B* is the null form ℓ_0 defined by $\ell_0(x) = 0 \quad \forall x \in B \Rightarrow \|\ell_0\| = 0$

Conversely, let $\|\ell\| = 0$. Since the set of $x \in B$ with $\|x\| = 1$ spans B, ℓ must equal ℓ_0.

$\Rightarrow \|\ell\| = 0$ iff $\ell = \ell_0$. 1pt

(ii) $\|\ell_1 + \ell_2\| = \sup_{\|x\|=1} \{|\ell_1(x) + \ell_2(x)|\} \leq \sup_{\|x\|=1} \{|\ell_1(x)| + |\ell_2(x)|\} = \|\ell_1\| + \|\ell_2\|$
That is, B* inherits the triangle inequality from $\mathbb{R}$ or $\mathbb{C}$. 1pt

(iii) $\|a\,\ell\| = \sup_{\|x\|=1} \{|a\,\ell(x)|\} = \sup_{\|x\|=1} \{|a|\,|\ell(x)|\} = |a| \sup_{\|x\|=1} \{\ell(x)\} = |a|\ \|\ell\|$
$\forall a \in \mathbb{R}$ or $\mathbb{C}, \ell \in$ B* 0.5 pt

$\Rightarrow$ The sup norm is indeed a norm