I.5.1. Lorentz transformations in M_2

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

Solution

a) Let $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For D to be a Lorentz transformation, we must have

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & ac - bd \\ ac - bd & c^2 - d^2 \end{pmatrix}$$

From this we obtain three constraints for the four numbers a, b, c, d:

(i) $a^2 - b^2 = 1$

(ii) $c^2 - d^2 = -1$

(iii) $ac - bd = 0$ 1pt

Now consider $\sinh\phi$, which maps $\mathbb{R}$ one-to-one onto itself. Therefore,

$\forall b \in \mathbb{R}\ \exists!\ \phi \in \mathbb{R}$ such that $b = \sinh\phi$

(i) $\Rightarrow a^2 = 1 + b^2 = 1 + \sinh^2\phi = \cosh^2\phi \Rightarrow a = \sigma\ \cosh\phi\ ,\ \sigma = \pm 1$

Analogously, $c = \sinh\psi$

(ii) $\Rightarrow d^2 = 1 + c^2 = 1 + \sinh^2\psi = \cosh^2\psi \Rightarrow d = \tau\ \cosh\psi\ ,\ \tau = \pm 1$ 1pt

Finally,

(iii) $\Rightarrow$

$$\begin{aligned} 0 = &\ \sigma\ \cosh\phi\ \sinh\psi - \tau\ \cosh\psi\ \sinh\phi \\ = &\ \cosh\phi\ \sinh\psi - \sigma\,\tau\ \cosh\psi\ \sinh\phi \\ = &\ \cosh(\sigma\tau\phi)\ \sinh\psi - \cosh\psi\ \sinh(\sigma\tau\phi) \\ = &\ \sinh(\psi - \sigma\tau\phi) \end{aligned}$$

$\Rightarrow \psi = \sigma\tau\phi$ 1pt

Therefore,

$$
\begin{aligned}
D_{\sigma,\tau}(\phi) &= \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sinh(\sigma\tau\phi) & \tau\cosh(\sigma\tau\phi) \end{pmatrix} = \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\tau\sinh(\phi) & \tau\cosh(\phi) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \sigma\cosh\phi & \sinh\phi \\ \sigma\sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} D(\phi) \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

with

$$D(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

where $\phi \in \mathbb{R}$ and $\sigma, \tau = \pm 1$, is the most general element of $O(1,1)$. 1pt

b) We have

(i)

$$\begin{pmatrix} \cosh\phi_1 & \sinh\phi_1 \\ \sinh\phi_1 & \cosh\phi_1 \end{pmatrix} \begin{pmatrix} \cosh\phi_2 & \sinh\phi_2 \\ \sinh\phi_2 & \cosh\phi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$$

$\Rightarrow D(\phi_1)D(\phi_2) = D(\phi_1+\phi_2) \Rightarrow$ closure ✓

(ii) Matrix multiplication is associative

(iii) $D(\phi = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$ is the neutral element

(iv) $D(-\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$

and $D(-\phi)D(\phi) = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow D(-\phi)$ is the inverse of $D(\phi)$

$\Rightarrow \{D(\phi)\}$ is a group $SO^+(1,1)$ under matrix multiplication,

and (i), (iii), (iv) provide an isomorphism $SO^+(1,1) \cong \mathbb{R}(+)$ 1pt

c) $e^{J\phi} = \mathbb{1}_2 + J\phi + \frac{1}{2}J^2\phi^2 + \ldots$

and $\cosh\phi = 1 + \frac{1}{2}\phi^2 + \frac{1}{4!}\phi^4 + \ldots$, $\sinh\phi = \phi + \frac{1}{3!}\phi^3 + \ldots$

Try $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad J^2 = \mathbb{1}_2$, $J^3 = J$ etc.

$$\Rightarrow e^{J\phi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \phi \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} \phi^2/2 & 0 \\ 0 & \phi^2/2 \end{pmatrix} + \begin{pmatrix} 0 & \phi^3/3! \\ \phi^3/3! & 0 \end{pmatrix} + \ldots = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

$\Rightarrow J$ is the generator of $SO^+(1,1)$ 1pt

I.5.3. Special Lorentz transformations in M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu$ (rotations)

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu$ (Lorentz boost along the x-direction)

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu$ (parity)

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu$ (time reversal)

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

Solution

I.5.3 a) i) $R^\mu{}_\alpha g_{\mu\nu} R^\nu{}_\lambda = \begin{pmatrix}1&0\\0&R^T\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&R\end{pmatrix} = \begin{pmatrix}1&0\\0&R^T\end{pmatrix}\begin{pmatrix}1&0\\0&-R\end{pmatrix}$

$= \begin{pmatrix}1&0\\0&-R^TR\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = g_{\alpha\lambda}$ ✓

ii) $B^\mu{}_\alpha g_{\mu\nu} B^\nu{}_\lambda = \begin{pmatrix}\cosh\alpha&0&0&\sinh\alpha\\0&1&0&0\\0&0&1&0\\\sinh\alpha&0&0&\cosh\alpha\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$

$\times\begin{pmatrix}\cosh\alpha&0&0&\sinh\alpha\\0&1&0&0\\0&0&1&0\\\sinh\alpha&0&0&\cosh\alpha\end{pmatrix}$

$= \begin{pmatrix}\cosh^2\alpha-\sinh^2\alpha&0&0&\cosh\alpha\sinh\alpha-\sinh\alpha\cosh\alpha\\0&-1&0&0\\0&0&-1&0\\\sinh\alpha\cosh\alpha-\cosh\alpha\sinh\alpha&0&0&\sinh^2\alpha-\cosh^2\alpha\end{pmatrix}$

(1) $= \begin{pmatrix}1&&&0\\&-1&&\\&&-1&\\0&&&-1\end{pmatrix} = g_{\alpha\lambda}$ ✓

iii) $P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = g_{\alpha\lambda}$ ✓

(1) iv) $T^\mu{}_\alpha g_{\mu\nu} T^\nu{}_\lambda = (-)^2 P^\mu{}_\alpha g_{\mu\nu} P^\nu{}_\lambda = g_{\alpha\lambda}$ ✓

(1) b) $R^\mu{}_\nu$ leaves the time coordinates invariant, and we know that the $R^i{}_j$ form a group (see PHYS 611) → $\{R^i{}_j\}$ is a subgroup of L. The boosts $B^\mu{}_\nu$ are a subgroup of L according to Problem 1.8.

(1) c) group tables:

	1	P
1	1	P
P	P	1

and

	1	T
1	1	T
T	T	1

→ $\{1, P\}$, and $\{1, T\}$ are subgroups of L under matrix multiplication

	1	P	T	PT
1	1	P	T	PT
P	P	1	PT	T
T	T	PT	1	P
PT	PT	T	P	1

→ $\{1, P, T, PT\}$ is also a subgroup of L

I.6.1. Transformations of tensor fields

a) Consider a covariant rank-n tensor field $t_{i_1 \ldots i_n}(x)$ and find its transformation law under normal coordinate transformations that is analogous to §5.1 def.1; i.e., find how $\tilde{t}_{i_1 \ldots i_n}(\tilde{x})$ is related to $t_{i_1 \ldots i_n}(x)$.

b) Convince yourself that your result is consistent with the transformation properties of (i) a covector x_i (the case $n = 1$), and (ii) the covariant components of the metric tensor g_{ij}.

(4 points)

Solution

I.6.1 a) There are various ways to do this. One option is to start with the transformation property of contravariant tensor fields, §5.1, and use the metric tensor to lower the indices:

$$\underline{\underline{\tilde t_{i_1\dots i_N}(\tilde x)}} = \tilde g_{i_1 j_1}\cdots \tilde g_{i_N j_N}\, \tilde t^{\,j_1\dots j_N}(\tilde x)$$

$$\overset{\S 5.1(a)+\tilde g = g}{=} g_{i_1 j_1}\cdots g_{i_N j_N}\, D^{j_1}{}_{\lambda_1}\cdots D^{j_N}{}_{\lambda_N}\, t^{\lambda_1\dots\lambda_N}(x)$$

$$= g_{i_1 j_1}\cdots g_{i_N j_N}\, D^{j_1}{}_{\lambda_1}\cdots D^{j_N}{}_{\lambda_N}\, g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\, t_{l_1\dots l_N}(x)$$

$$= (gD)_{i_1\lambda_1}\cdots (gD)_{i_N\lambda_N}\, g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\, t_{l_1\dots l_N}(x)$$

$$\overset{gD = (D^T)^{-1} g}{=} \left((D^T)^{-1} g\right)_{i_1\lambda_1}\cdots \left((D^T)^{-1} g\right)_{i_N\lambda_N}\, g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\, t_{l_1\dots l_N}(x)$$

$$= \left((D^T)^{-1}\right)_{i_1}{}^{m_1} g_{m_1\lambda_1}\cdots \left((D^T)^{-1}\right)_{i_N}{}^{m_N} g_{m_N\lambda_N}\, g^{\lambda_1 l_1}\cdots g^{\lambda_N l_N}\, t_{l_1\dots l_N}$$

$$= \left((D^T)^{-1}\right)_{i_1}{}^{m_1} \underbrace{g_{m_1\lambda_1} g^{\lambda_1 l_1}}_{=\delta_{m_1}{}^{l_1}}\cdots \left((D^T)^{-1}\right)_{i_N}{}^{m_N} \underbrace{g_{m_N\lambda_N} g^{\lambda_N l_N}}_{=\delta_{m_N}{}^{l_N}}\, t_{l_1\dots l_N}(x)$$

(2) $$\underline{\underline{= \left((D^T)^{-1}\right)_{i_1}{}^{j_1}\cdots \left((D^T)^{-1}\right)_{i_N}{}^{j_N}\, t_{j_1\dots j_N}(x)}}$$

This is §5.1 (a) with $D^{i_n}{}_{j_n}$ replaced by $\left((D^T)^{-1}\right)_{i_n}{}^{j_n}$ $(n=1,\dots,N)$

b) Special case $n=1$:

(1) $$\underline{\underline{\tilde x_i}} = \tilde g_{ij}\,\tilde x^j \overset{\S 4.8.2}{=} \tilde g_{ij}\, D^j{}_\lambda\, x^\lambda \overset{\tilde g = g}{=} (gD)_{i\lambda}\, x^\lambda \overset{gD = D^{T-1} g}{=} \left((D^T)^{-1} g\right)_{i\lambda} x^\lambda$$

$$= \left((D^T)^{-1}\right)_i{}^j\, g_{j\lambda}\, x^\lambda = \underline{\underline{\left((D^T)^{-1}\right)_i{}^j}}\, x_j \quad ✓$$

(1) $$\tilde g_{ij} = g_{ik}\, g_{jl}\, \tilde g^{kl} \overset{\S 5.1}{=} g_{ik}\, g_{jl}\, D^k{}_m D^l{}_n\, g^{mn} = (gD)_{im} (gD)_{jn}\, g^{mo} g^{np} g_{op}$$

$$= (D^{T-1} g)_{im}\, g^{mo}\, (D^{T-1} g)_{jn}\, g^{np}\, g_{op} = \left((D^T)^{-1}\right)_i{}^o \left((D^T)^{-1}\right)_j{}^p\, g_{op} \quad ✓$$

I.6.2. Curl and divergence

Show that the curl and the divergence of a vector field transform as a pseudovector field and a scalar field, respectively.

(3 points)

Solution

I.6.2.) Consider the curl as defined in § 5.2 :

$$c^i(x) = \varepsilon^{ijk} \partial_j v_k(x)$$

$$\to \tilde{c}^i(\tilde{x}) = \tilde{\varepsilon}^{ijk} \tilde{\partial}_j \tilde{v}_k(\tilde{x})$$

Prob. 2.5

$$= \tilde{\varepsilon}^{ijk} (D^{-1})^l{}_j \partial_l (D^{-1})^m{}_k v_m(x)$$

$$= \delta^i{}_n \tilde{\varepsilon}^{njk} (D^{-1})^l{}_j (D^{-1})^m{}_k \partial_l v_m(x)$$

$$= D^i{}_p (D^{-1})^p{}_n \tilde{\varepsilon}^{njk} (D^{-1})^l{}_j (D^{-1})^m{}_k \partial_l v_m(x)$$

$$= D^i{}_p \underbrace{(D^{-1})^p{}_n (D^{-1})^l{}_j (D^{-1})^m{}_k \tilde{\varepsilon}^{njk}}_{= (\det D^{-1})\, \varepsilon^{plm} \text{ by § 5.1 remark (9)}} \partial_l v_m(x)$$

$$= (\det D)\, D^i{}_p \underbrace{\varepsilon^{plm} \partial_l v_m(x)}_{= c^p(x)}$$

(2) $$= (\det D)\, D^i{}_p\, c^p(x)$$

$\to$ $c^i(x)$ transforms as a pseudovector field.

Now the divergence: $d(x) = \partial_i v^i(x)$

$$\to \tilde{d}(\tilde{x}) = \tilde{\partial}_i \tilde{v}^i(\tilde{x}) = ((D^T)^{-1})_i{}^j \partial_j D^i{}_k v^k(x)$$

$$= \underbrace{(D^T)_k{}^i ((D^T)^{-1})_i{}^j}_{= \delta_k{}^j} \partial_j v^k(x)$$

(1) $$= \partial_k v^k(x)$$ $\to$ $d(x)$ transforms as a scalar field