

Problem Assignment # 7

11/13/2024
due 11/20/2024

II.2.1. Lindhard function

Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ (which plays an important role in the theory of many-electron systems) defined by

$$f(z) = \log \left(\frac{z-1}{z+1} \right)$$

The *spectrum* $f'' : \mathbb{R} \rightarrow \mathbb{R}$ and the *reactive part* $f' : \mathbb{R} \rightarrow \mathbb{R}$ of f are defined by

$$f''(\omega) := \frac{1}{2i} [f(\omega + i0) - f(\omega - i0)] \quad , \quad f'(\omega) := \frac{1}{2} [f(\omega + i0) + f(\omega - i0)]$$

where $f(\omega \pm i0) := \lim_{\epsilon \rightarrow 0} f(\omega \pm i\epsilon)$.

- Show that f' and f'' are indeed real-valued functions.
- Determine f'' and f' explicitly, and plot them for $-3 < \omega < 3$.
- Show that

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{f''(\omega)}{\omega - z} = f(z)$$

(5 points)

II.2.2. Another causal function

The function considered in Problem 2.2.1 is an example of a class of complex functions called *causal functions* that are important in the theory of many-particle systems. Another member of this class is

$$g(z) = \sqrt{z^2 - 1} - z$$

Determine the spectrum and the reactive part of $g(z)$, and plot them for $-3 < \omega < 3$.

(3 points)

II.2.3. Proof of the Cauchy-Riemann Theorem

Prove the Cauchy-Riemann theorem from ch.2 §2.2:

- Let $f(z) = f'(z', z'') + i f''(z', z'')$ be analytic everywhere in $\Omega \subseteq \mathbb{C}$. Show that the Cauchy-Riemann equations

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \quad \text{and} \quad \frac{\partial f'}{\partial z''} = - \frac{\partial f''}{\partial z'}$$

hold $\forall z \in \Omega$.

hint: Start with the difference quotient $(f(z) - f(z_0))/(z - z_0)$ and require that its limit for $z \rightarrow z_0$ exists if z_0 is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

- Let the Cauchy-Riemann equations hold in a point $z_0 \in \Omega$. Show that this implies that f is analytic in the point z_0 .

hint: Consider $f(z) - f(z_0)$ and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series about z_0 .

(8 points)

II.2.4. Exponentials

Consider the exponential function

$$f(z) = e^z = e^{z'+iz''}$$

- a) Show that $f(z)$ is analytic everywhere in \mathbb{C} .
- b) Convince yourself explicitly that the real and imaginary parts of f obey Laplace's differential equation.
- c) Show that $df/dz|_z = f(z)$.
- d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \quad , \quad \sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$$

are analytic everywhere in \mathbb{C} , and that

$$\frac{d}{dz} \cos z = -\sin z \quad , \quad \frac{d}{dz} \sin z = \cos z .$$

(4 points)

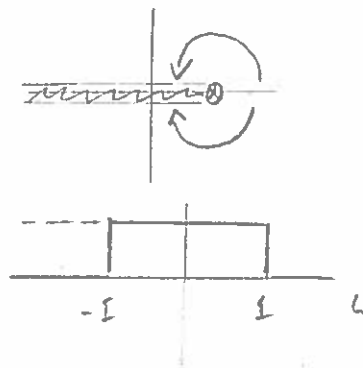
2.2.1) a) $f(u-i0) = f(u+i0)^*$, od $z+z^* = z'+iz''+z'-iz'' = 2z' \in \mathbb{R}$
 od $\frac{1}{i}(z-z^*) = 2z'' \in \mathbb{R}$

$\Rightarrow f'(u), f''(u) \in \mathbb{R}$

b) $f(z) = \log \frac{z-1}{z+1} = \log(z-1) - \log(z+1)$

$\Rightarrow f''(u) = \frac{1}{zi} [\log(u-1+i0) - \log(u-1-i0) - \log(u+1+i0) + \log(u+1-i0)]$

$\Rightarrow \underline{zi f''(u)} = \theta(u \leq -1) [i\pi - (-i\pi)]$
 $- \theta(u < -1) [i\pi - (-i\pi)]$
 $= (\theta(u < 1) - \theta(u < -1)) 2i\pi$
 $= \underline{\theta(u^2 < 1) 2i\pi}$

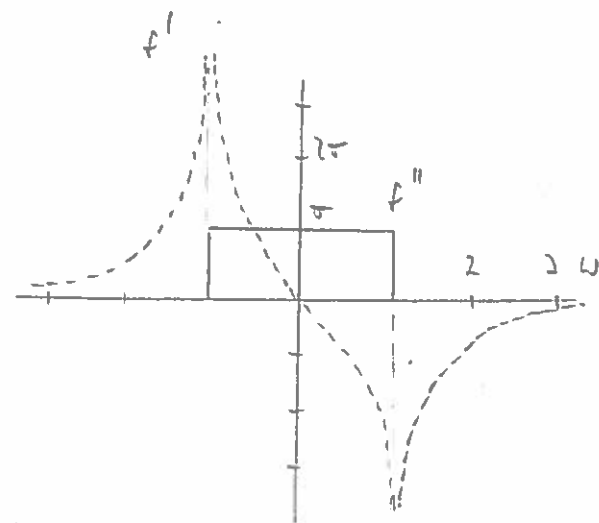


$\Rightarrow \underline{\underline{f''(u) = \pi \theta(u^2 < 1)}}$

$f'(u) = \frac{1}{z} [\log(u-1+i0) + \log(u-1-i0) - \log(u+1+i0) - \log(u+1-i0)]$

$\Rightarrow \underline{zf'(u) = 2 \log|u-1| - 2 \log|u+1|}$

$\Rightarrow \underline{\underline{f'(u) = \log \left| \frac{u-1}{u+1} \right|}}$



c) $\int_{-\infty}^{\infty} \frac{d\omega}{\sigma} \frac{f'(\omega)}{\omega-z} =$

$= \int_{-1}^1 d\omega \frac{1}{\omega-z} = \log(\omega-z) \Big|_{-1}^1$

$= \log(1-z) - \log(-1-z) = \log \frac{1-z}{-1-z}$

$= \log \frac{z-1}{z+1} = \underline{\underline{f(z)}}$

2.2.2.)

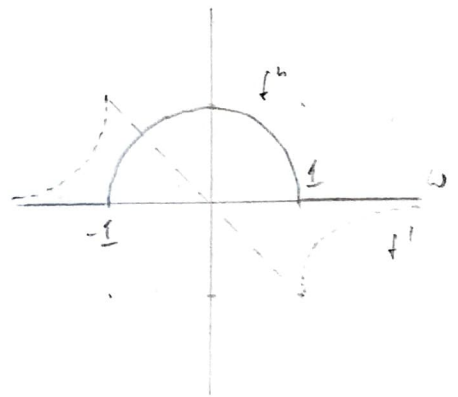
$$f(z) = \sqrt{z^2 - 1} - z = \sqrt{z+1} \sqrt{z-1} - z$$

$$\begin{aligned} \underline{f''(w)} &= \frac{1}{i} \left[\sqrt{w+1+i0} \sqrt{w-1+i0} - \sqrt{w+1-i0} \sqrt{w-1-i0} \right] - \frac{1}{i} (w+i0 - w-i0) \\ &= \frac{1}{i} \left[\left(\Theta(w>-1) (w+1)^{1/2} + \Theta(w<-1) |w+1|^{1/2} i \right) \left(\Theta(w>1) (w-1)^{1/2} + \Theta(w<1) |w-1|^{1/2} i \right) - w \right] \\ &= \Theta(w>-1) \Theta(w<1) |w^2-1|^{1/2} + \underbrace{\Theta(w<-1) \Theta(w>1) |w^2-1|^{1/2}}_{=0} \\ &= \underline{\underline{\Theta(w^2 < 1) (1-w^2)^{1/2}}} \end{aligned}$$

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$$\begin{aligned} \underline{f'(w)} &= \frac{1}{i} [f(w+i0) + f(w-i0)] = i \operatorname{Re} f(w+i0) \\ &= \Theta(w>-1) \Theta(w>1) \sqrt{w^2-1} - \Theta(w<-1) \Theta(w>1) \sqrt{w^2-1} - w \\ &= \underline{\underline{\Theta(w^2 < 1) (-w) + \Theta(w^2 > 1) (\sqrt{w^2-1} \operatorname{sgn} w - w)}} \end{aligned}$$

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2.2.2.1a) Consider the difference quotient

$$\frac{\Delta f}{\Delta z} = \frac{f'(z', z'') + i f''(z', z'') - f'(z_0', z_0'') - i f''(z_0', z_0'')}{z' + i z'' - z_0' - i z_0''}$$

for f in the vicinity of a point $z_0 \in \mathbb{R}$.

For the limit $\frac{df}{dz} = \lim_{z \rightarrow z_0} \frac{\Delta f}{\Delta z}$ to exist, it must exist

for f' and f'' separately, and it must exist for $z \rightarrow z_0$ along any path. In particular, it must exist for $z \rightarrow z_0$ along the real axis and along the imaginary axis.

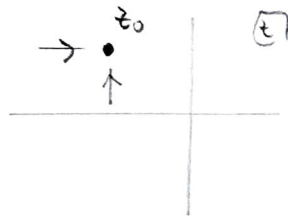
\rightarrow If f is differentiable in z_0 , then the derivatives

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$\frac{\partial f'}{\partial z'}, \frac{\partial f'}{\partial z''}, \frac{\partial f''}{\partial z'}, \frac{\partial f''}{\partial z''}$ all must exist in z_0

Now approach z_0 parallel to the real axis, i.e., for fixed z'' :

$$\left. \frac{df}{dz} \right|_{z_0} = \left. \frac{\partial f'}{\partial z'} \right|_{z_0} + i \left. \frac{\partial f''}{\partial z'} \right|_{z_0} \quad (*)$$



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Now approach for fixed z' :

$$\left. \frac{df}{dz} \right|_{z_0} = i \left. \frac{\partial f''}{\partial z''} \right|_{z_0} + \left. \frac{\partial f'}{\partial z''} \right|_{z_0} = \left. \frac{\partial f''}{\partial z''} \right|_{z_0} - i \left. \frac{\partial f'}{\partial z''} \right|_{z_0} \quad (**)$$

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That if f is differentiable in z_0 , then $(*) = (**)$

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$$\Rightarrow \boxed{\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''}} \text{ and } \boxed{\frac{\partial f''}{\partial z'} = -\frac{\partial f'}{\partial z''}} \rightarrow \text{The Cauchy-Riemann eqs are necessary}$$

b) Consider the numerator of the difference quotient,

$$f(z) - f(z_0) = f'(z', z'') + i f''(z', z'') - f'(z_0', z_0'') - i f''(z_0', z_0'')$$

and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series:

$$f'(z', z'') - f'(z_0', z_0'') = \frac{\partial f'}{\partial z'} \Big|_{z_0} (z' - z_0') + \frac{\partial f'}{\partial z''} \Big|_{z_0} (z'' - z_0'') + \dots$$

$$f''(z', z'') - f''(z_0', z_0'') = \frac{\partial f''}{\partial z'} \Big|_{z_0} (z' - z_0') + \frac{\partial f''}{\partial z''} \Big|_{z_0} (z'' - z_0'') + \dots$$

$$\Rightarrow \frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{z - z_0} \left[\frac{\partial f'}{\partial z'} \Big|_{z_0} (z' - z_0') + \frac{\partial f'}{\partial z''} \Big|_{z_0} (z'' - z_0'') + i \frac{\partial f''}{\partial z'} \Big|_{z_0} (z' - z_0') + i \frac{\partial f''}{\partial z''} \Big|_{z_0} (z'' - z_0'') \right]$$

(this last term vanishes as $z \rightarrow z_0$)

$$\stackrel{\text{CR-eps}}{\downarrow} = \frac{1}{z - z_0} \left[\frac{\partial f'}{\partial z'} \Big|_{z_0} (z' - z_0' + i z'' - i z_0'') + \frac{\partial f'}{\partial z''} \Big|_{z_0} (z'' - z_0'' - i z' + i z_0') \right]$$

$$= \frac{\frac{\partial f'}{\partial z'} \Big|_{z_0}}{z - z_0} \frac{z - z_0}{z - z_0} - i \frac{\frac{\partial f'}{\partial z''} \Big|_{z_0}}{z - z_0} \frac{z - z_0}{z - z_0}$$

$$= \left(\frac{\partial f'}{\partial z'} - i \frac{\partial f'}{\partial z''} \right) \Big|_{z_0} \stackrel{\text{CR-eps}}{=} \frac{\partial f'}{\partial z'} + i \frac{\partial f''}{\partial z'} + \text{(this last term vanishes as } z \rightarrow z_0 \text{)}$$

But our premise was that the CR-eps hold

\Rightarrow the rhs exists

\Rightarrow the limit $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{df}{dz} \Big|_{z_0}$ exists and is

independent of how z_0 is approached.

\Rightarrow The Cauchy-Riemann eqs are sufficient

$$2.2.4.) a) f(z) = e^z = e^{t+iz} = e^t e^{iz} = e^t (w e^t + i n e^t)$$

$$\rightarrow \underline{f'(t, t^h) = e^t w e^t}$$

$$\underline{f''(t, t^h) = e^t i e^t}$$

$$\rightarrow \underline{\frac{\partial f'}{\partial t'} = e^t w e^t = \frac{\partial f''}{\partial t^h}}$$

$$\underline{\frac{\partial f'}{\partial t^h} = -e^t i e^t = -\frac{\partial f''}{\partial t'}}$$

$\rightarrow f$ is analytic on \mathbb{C}

b) $\underline{\frac{\partial^2 f'}{\partial t'^2} + \frac{\partial^2 f'}{\partial t^h{}^2} = e^t w e^t - e^t w e^t = 0}$

$\underline{\frac{\partial^2 f''}{\partial t'^2} + \frac{\partial^2 f''}{\partial t^h{}^2} = e^t i e^t - e^t i e^t = 0}$

c) Approach to show the real axis

$$\rightarrow \underline{\frac{df}{dz}(z) = \frac{\partial f'}{\partial t'} \Big|_z + i \frac{\partial f''}{\partial t'} \Big|_z = e^t w e^t + i e^t i e^t = e^t (w e^t + i i e^t)}$$

$$= e^t e^{it} = e^z = \underline{f(z)} \quad \square$$

d) Wieder $w e^z = \frac{1}{2}(e^{iz} + e^{-iz})$

$$i n e^z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

e^z is analytic regular \rightarrow so is $e^{t+iz} \rightarrow$ so are $w e^z$ and $i n e^z$

$$\underline{\frac{d}{dz} w e^z = \frac{1}{2}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{iz} - e^{-iz}) = -i n e^z}$$

$$\underline{\frac{d}{dz} i n e^z = \frac{1}{2i}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{iz} + e^{-iz}) = w e^z}$$