PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 8 11/20/2024
due 11/27/2024

II.3.1. Laurent series

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter Theorem 2 in ch. 2 §3.2.

(3 points)

II.3.2. Applications of the residue theorem

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx \, \frac{1}{x^4 + 1}$$

b)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

c)

$$\int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \frac{1}{1 + x^2}$$

and check your results numerically..

Let $a \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Use the residue theorem to show that

d)

$$\int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

e)

$$\int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{x^2 + a^2}$$

and define its Cauchy principal value by

$$\lim_{R \to 0} [\int_{-\infty}^{-R} dx \, f(x) + \int_{R}^{\infty} dx \, f(x)]$$

with $f(x) = 1/x(x^2 + a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \to 0$.

(17 points)

II.3.3. Matsubara frequency sum

Let $f(z)$ have simple poles at z_j $(j = 1, 2, \ldots)$, and no other singularities. Let $f(|z| \to \infty)$ go to zero faster then $1/z$. Consider the infinite sum

$$S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$$

with $\Omega_n = 2\pi T n$ and $T > 0$. Show that

$$S = \sum_j n(z_j) \operatorname{Res} f(z_j)$$

where $n(z) = 1/(e^{z/T} - 1)$ is the Bose distribution function.

hint: Show that $n(z)$ has simple poles at $z = i\Omega_n$, and integrate $n(z)\, f(z)$ over an infinite circle centered on the origin.

note: Sums of this form are important in finite-temperature quantum field theory. In this context, T is the temperature and Ω_n is called a "bosonic Matsubara frequency".

(3 points)

II.3.1. Laurent series

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter the theorem in ch. 2 §3.2.

(3 points)

Solution:

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{z-i}\,\frac{1}{2i+(z-i)} = \frac{1}{z-i}\,\frac{1}{2i}\,\frac{1}{1+\frac{1}{2i}(z-i)}$$

1 pt

Now expand

$$\frac{1}{1+\frac{1}{2i}(z-i)} = \sum_{n=0}^{\infty}(-)^n\,\frac{(z-i)^n}{(2i)^n}$$

$$\Rightarrow \quad f(z) = \frac{1}{z-i}\,\frac{1}{2i}\sum_{n=0}^{\infty}(-)^n\,\frac{(z-i)^n}{(2i)^n} = \sum_{n=0}^{\infty}(-)^n\,\frac{(z-i)^{n-1}}{(2i)^{n+1}} = \sum_{n=-1}^{\infty}\frac{(-)^{n+1}}{(2i)^{n+2}}\,(z-i)^n$$

1 pt

$\Rightarrow$ The coefficients f_n in the Laurent series are

$$f_n = \begin{cases} (-)^{n+1}/(2i)^{n+2} & \text{for} \quad n \geq -1 \\ 0 & \text{for} \quad n < -1 \end{cases}$$

1 pt

II.3.2. **Applications of the residue theorem**

Use complex analysis to evaluate the real integrals

a)

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b)

$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

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$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x}\, \frac{1}{1+x^2}$$

and check your results by means of Wolfram Alpha.

Let $a \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Use the residue theorem to show that

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with $f(x) = 1/x(x^2+a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \to 0$.

(17 points)

Solution

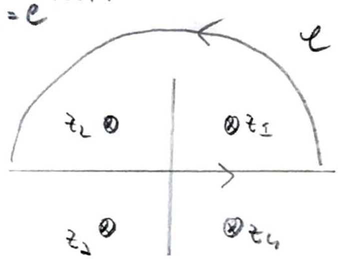
2.3.2.) a) $I = \int_{-\infty}^{\infty} dx \, \frac{1}{x^4+1}$

Consider $f(z) = \frac{1}{z^4+1} = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

poles: $z^4 = -1 \rightsquigarrow z_1 = e^{i\pi/4},\ z_2 = e^{3i\pi/4}$

$z_3 = e^{5i\pi/4},\ z_4 = e^{7i\pi/4}$

1



$\rightsquigarrow \underline{\underline{I}} = \int_{\mathcal{C}} dz \, f(z)$

$= 2\pi i \left[\text{Res}\, f(z_1) + \text{Res}\, f(z_2)\right]$

1

$\text{Res}\, f(z_1) = (z-z_1) f(z)\Big|_{z=z_1} = \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$

$= \frac{1}{\left(e^{i\frac{\pi}{4}}-e^{i\frac{3\pi}{4}}\right)\left(e^{i\frac{\pi}{4}}-e^{i\frac{5\pi}{4}}\right)\left(e^{i\frac{\pi}{4}}-e^{i\frac{7\pi}{4}}\right)} = \frac{e^{-i\pi/4}}{(1-i)(1+1)(1+i)}$

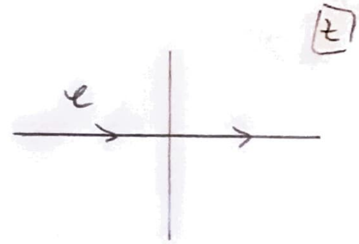
$= \underline{\frac{1}{4} e^{-i\pi/4}}$

$\text{Res}\, f(z_2) = \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{e^{-i3\pi/4}}{(1+i)(1-i)(1+1)} = \underline{\frac{1}{4} e^{-i3\pi/4}}$

$\rightsquigarrow \underline{\underline{I}} = 2\pi i \frac{1}{4} e^{-i\pi/4}\left(1 + e^{-i\pi/2}\right) = \frac{i\pi}{2} \frac{1}{\sqrt{2}} (1-i)^2$

$= \frac{i\pi}{2\sqrt{2}}(-2i) = \underline{\underline{\frac{\pi}{\sqrt{2}}}}$

1



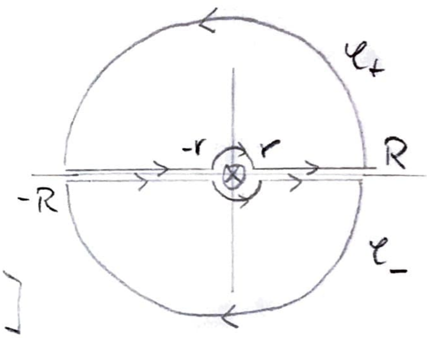
b) $\underline{\underline{J}} = \int dx \, \frac{\sin x}{x} = \int_{\mathcal{C}} dt \, \frac{\sin t}{t}$

$= \frac{1}{2i} \int_{\mathcal{C}} dt \, \frac{1}{t} \left[e^{it} - e^{-it} \right]$

remark: $\frac{\sin t}{t}$ is analytic everywhere

$\frac{1}{t} e^{\pm it}$ has a simple pole at $t=0$ with residue 1.

consider



$\tilde{J}_\pm(R,r) := \frac{1}{2i} \int_{\mathcal{C}_\pm} dt \, \frac{1}{t} e^{\pm it}$

$= \frac{1}{2i} \left[\int_{-R}^{r} dx \, \frac{1}{x} e^{\pm ix} + \int_{r}^{R} dx \, \frac{1}{x} e^{\pm ix} \right]$

$+ \frac{1}{2i} \int_{c_\pm} dt \, \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_\pm} dt \, \frac{1}{t} e^{\pm it}$

$=: J_\pm(R,r) + \frac{1}{2i} \int_{c_\pm} dt \, \frac{1}{t} e^{\pm it} + \frac{1}{2i} \int_{C_\pm} dt \, \frac{1}{t} e^{\pm it}$ (1)

where

$J_\pm(R,r) = \int_{-R}^{-r} dx \, \frac{1}{x} e^{\pm ix} + \int_{r}^{R} dx \, \frac{1}{x} e^{\pm ix}$

and $c_\pm$ = semicircle with radius r

$C_\pm$ = semicircle with radius R

$\rightsquigarrow \underline{\underline{J}} = \lim_{\substack{r \to 0 \\ R \to \infty}} \left[J_+(R,r) - J_-(R,r) \right]$

(1)

Parametrization

$$\left.\begin{aligned} C_+ &: t = r e^{i\varphi} \quad (\pi \le \varphi \le 0) \\ C_- &: t = r e^{i\varphi} \quad (-\pi \le \varphi \le 0) \end{aligned}\right\} \; dt = i r e^{i\varphi} d\varphi$$

$$\left.\begin{aligned} C'_+ &: t = R e^{i\varphi} \quad (0 \le \varphi \le \pi) \\ C'_- &: t = R e^{i\varphi} \quad (0 \ge \varphi \ge -\pi) \end{aligned}\right\} \; dt = i R e^{i\varphi} d\varphi$$

$$\rightsquigarrow \underline{\int_{C_+} dt \, \frac{1}{t} e^{it}} = ir \int_\pi^0 d\varphi \, \cancel{e^{i\varphi}} \frac{1}{r\cancel{e^{i\varphi}}} e^{ire^{i\varphi}} = -i \int_0^\pi d\varphi \, e^{ire^{i\varphi}}$$

$$\stackrel{r\to 0}{=} \underline{-i\pi + O(r)}$$

$$\underline{\int_{C_-} dt \, \frac{1}{t} e^{-it}} = ir \int_{-\pi}^0 d\varphi \, \cancel{e^{i\varphi}} \frac{1}{r\cancel{e^{i\varphi}}} e^{-ire^{i\varphi}} = \underline{i\pi + O(r)}$$

1

$$\underline{\int_{C'_+} dt \, \frac{1}{t} e^{it}} = iR \int_0^\pi d\varphi \, \cancel{e^{i\varphi}} \frac{1}{R\cancel{e^{i\varphi}}} e^{iR(\cos\varphi + i\sin\varphi)} = i \int_0^\pi d\varphi \, e^{iR\cos\varphi - R\sin\varphi}$$

$$\underline{\int_{C'_-} dt \, \frac{1}{t} e^{-it}} = -iR \int_{-\pi}^0 d\varphi \, \cancel{e^{i\varphi}} \frac{1}{R\cancel{e^{i\varphi}}} e^{iR\cos\varphi + R\sin\varphi} = i \int_0^\pi d\varphi \, e^{iR\cos\varphi - R\sin\varphi}$$

$$\rightsquigarrow \underline{\left| \int_{C'_\pm} dt \, \frac{1}{t} e^{\pm it} \right|} = \left| \int_0^\pi d\varphi \, e^{iR\cos\varphi - R\sin\varphi} \right| \le \int_0^\pi d\varphi \, \underbrace{\left| e^{iR\cos\varphi} \right|}_{=1} \cdot e^{-R\sin\varphi}$$

$$= 2 \int_0^{\pi/2} d\varphi \, e^{-R\sin\varphi} = 2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} e^{-Rx}$$

$$= \frac{2}{R} \int_0^R dx \, \frac{1}{\sqrt{1-x^2/R^2}} e^{-x} = \frac{2}{R} \left[\int_0^\infty dx \, e^{-x} + O(e^{-R}) \right]$$

$$\underline{= \frac{2}{R} \left[1 + O(e^{-R}) \right] \to 0 \quad \text{for } R \to \infty}$$

1

Now, $\tilde{\gamma}_\pm(R, r) = 0$ by the residue theorem

$$\rightarrow \underline{\underline{I}} = \lim_{\substack{r \to 0 \\ R \to \infty}} \left[J_+(R,r) - J_-(R,r) \right]$$

$$= \lim_{\substack{r \to 0 \\ R \to \infty}} \Big[\tilde{J}_+(R,r) - \frac{1}{2i} \int_{C_+} dt \, \frac{1}{t} e^{it} - \frac{1}{2i} \int_{C_+} dt \, \frac{1}{t} e^{it}$$

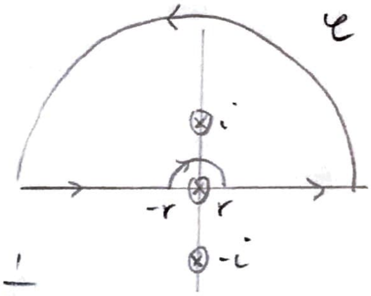
$$- \tilde{J}_-(R,r) + \frac{1}{2i} \int_{C_-} dt \, \frac{1}{t} e^{-it} + \frac{1}{2i} \int_{C_-} dt \, \frac{1}{t} e^{-it} \Big]$$

(1) $$= 0 - \frac{1}{2i}(-i\pi) - 0 - 0 + \frac{1}{2i} i\pi + 0 = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$

c) $\underline{\underline{I}} = \int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \, \frac{1}{x^2+1} = \int_{-\infty}^{\infty} dx \, \frac{1}{x} \, \frac{1}{2i} \left(e^{ix} - e^{-ix}\right) \frac{1}{x^2+1}$

$= -i \int_{-\infty}^{\infty} dx \, \underbrace{\frac{e^{ix}}{x} \, \frac{1}{x^2+1}} = -i \int_{-\infty}^{\infty} dx \, \frac{e^{ix}}{x} \, \frac{1}{(x+i)(x-i)}$

$= -i \int_{-\infty}^{\infty} dx \; f(x)$



(1) $f(|z| \to 0)$ for $\operatorname{Im} z \geq 0$

$\rightsquigarrow \int_{\mathcal{C}} dz \, f(z) = 2\pi i \; \operatorname{Res} f(z=i) = 2\pi i \, \frac{e^{-1}}{i} \, \frac{1}{2i}$

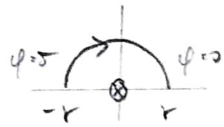
$= -i\pi \frac{1}{e}$

"

$\int_{-\infty}^{\infty} dx \, f(x) + \lim_{r \to 0} \int_{\frown} dz \, f(z)$

$\rightsquigarrow \int_{-\infty}^{\infty} dx \, f(x) = -\frac{i\pi}{e} - \lim_{r \to 0} \int_{\frown} dz \, f(z)$

(1)



Along the semicircle, $z = r(\cos\varphi + i \sin\varphi)$

with $r = \text{const.}$

$\rightsquigarrow \; dz = r(-\sin\varphi + i\cos\varphi)\, d\varphi = ir(\cos\varphi + i\sin\varphi)\, d\varphi$

$\rightsquigarrow \int_{\frown} dz \, f(z) = i\cancel{r} \int_{\pi}^{0} d\varphi \, \cancel{(\cos\varphi + i\sin\varphi)} \, \frac{e^{ir(\cos\varphi + i\sin\varphi)}}{\cancel{r}\cancel{(\cos\varphi + i\sin\varphi)}} \left[1 + O(r^2)\right]$

$= -i\pi + O(r)$

$\rightsquigarrow \int_{-\infty}^{\infty} dx \, f(x) = \frac{-i\pi}{e} + i\pi = \frac{i\pi}{e}(e-1)$

(1) $\rightsquigarrow \; \underline{\underline{I = \frac{\pi}{e}(e-1)}}$

p 2.3.2-6

WolframAlpha computational knowledge engine

integral

☰ Examples ⤮ Random

Assuming "integral" refers to a computation | Use as a general topic or a character or referring to a mathematical definition or a word instead

- function to integrate: sin x/(x(1+x^2))
- lower limit: -Infinity
- upper limit: Infinity

Also include: variable

Definite integral: More digits

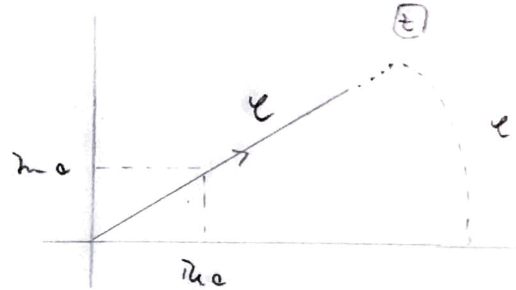
$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x\left(1+x^2\right)}\, dx = \frac{(e-1)\,\pi}{e} \approx 1.98587$$

Computed by Wolfram *Mathematica*

Download as: PDF | Live *Mathematica*

d) $I := \int_{-\infty}^{\infty} dx\, e^{-ax^2} = 2\int_{0}^{\infty} dx\, e^{-ax^2}$ $\qquad \underline{\mathrm{Re}\, a > 0}$

$z = \sqrt{a}\, x \,, \quad dz = \sqrt{a}\, dx$



1 $\quad \to \underline{I = \frac{2}{\sqrt{a}} \int_{C} dz\, e^{-z^2}}$

On the other hand, the residue theorem yields

1 $\quad \int_{C} dz\, e^{-z^2} + \int_{C'} dz\, e^{-z^2} + \int_{\infty}^{0} dz\, e^{-z^2} = 0$

$$\to \underline{\tfrac{1}{2}\sqrt{a}\, I} = \int_{C} dz\, e^{-z^2} = -\underbrace{\int_{C'} dz\, e^{-z^2}}_{=0} + \int_{0}^{\infty} dz\, e^{-z^2} = \int_{0}^{\infty} dx\, e^{-x^2}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} dx\, e^{-x^2} = \frac{1}{2}\left[\int_{-\infty}^{\infty} dx\, dy\, e^{-(x^2+y^2)}\right]^{1/2}$$

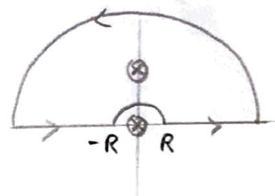
$$= \frac{1}{2}\left(\int_{0}^{2\pi} d\varphi \int_{0}^{\infty} dr\, r\, e^{-r^2}\right)^{1/2} = \frac{1}{2}\left(2\pi\, \frac{1}{2} \underbrace{\int_{0}^{\infty} dx\, e^{-x}}_{=1}\right)^{1/2} = \underline{\frac{\sqrt{\pi}}{2}}$$

1 $\quad \to \underline{\underline{I = \sqrt{\pi/a}}}$

c) $\underline{\mathcal{I}} = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{a^2+x^2} = \underline{\frac{1}{a^2}\mathcal{J}}$,

$$\mathcal{J} = \int_{-\infty}^{\infty} \frac{dx}{x} \frac{1}{1+x^2} = \int_{-\infty}^{\infty} \frac{dz}{z} \frac{1}{1+z^2}$$

remark: This integral exists only in the sense of a principal value



1 $\underline{\mathcal{J}} = \lim_{R\to 0} \int_{C} \frac{dz}{z} \frac{1}{1+z^2} - \lim_{R\to 0} \int_{\frown} \frac{dz}{z} \frac{1}{1+z^2}$

poles: $z=0$, $z=\pm i$

$$\mathrm{Res}\left(\frac{1}{z}\frac{1}{1+z^2}, z=i\right) = \frac{1}{i}\frac{1}{2i} = \frac{-1}{2}$$

$$\Rightarrow \underline{\mathcal{J}} = 2\pi i \frac{-1}{2} - \lim_{R\to 0} \int_{\pi}^{0} d\theta \, iR \frac{e^{i\theta}}{Re^{i\theta}} \frac{1}{1+R^2 e^{2i\theta}} = -\pi i - i \int_{\pi}^{0} d\theta$$

$$= -\pi i - i(-\pi) = \underline{0}$$

1 $\Rightarrow \underline{\mathcal{I} = 0}$

remark: This result is consistent with the usual principal value for an antisymmetric integrand:

$$\underline{\mathrm{P.V.}(\mathcal{I})} = \lim_{R\to 0} \left[\int_{-\infty}^{-R} \frac{dx}{x} \frac{1}{a^2+x^2} + \int_{R}^{\infty} \frac{dx}{x} \frac{1}{a^2+x^2} \right] =$$

$$= \lim_{R\to 0} \int_{R}^{\infty} dx \left[\frac{1}{x}\frac{1}{a^2+x^2} - \frac{1}{x}\frac{1}{a^2+x^2} \right] = \underline{0}$$

1

II.3.3. Matsubara frequency sum

Let $f(z)$ have simple poles at z_j $(j = 1, 2, \ldots)$, and no other singularities. Let $f(|z| \to \infty)$ go to zero faster then $1/z$. Consider the infinite sum

$$S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$$

with $\Omega_n = 2\pi T n$ and $T > 0$. Show that

$$S = \sum_j n(z_j) \, \mathrm{Res}\, f(z_j)$$

where $n(z) = 1/(e^{z/T} - 1)$ is the Bose distribution function.

hint: Show that $n(z)$ has simple poles at $z = i\Omega_n$, and integrate $n(z)\, f(z)$ over an infinite circle centered on the origin.

note: Sums of this form are important in finite-temperature quantum field theory. In this context, T is the temperature and Ω_n is called a "bosonic Matsubara frequency".

(3 points)

Solution

2.2.2.) $S = -T \sum_{n=-\infty}^{\infty} f(i\Omega_n)$ with $\Omega_n = 2\pi T n$ $(n \in \mathbb{Z})$

Consider $n(z) = \dfrac{1}{e^{z/T}-1}$

$$\rightsquigarrow n(i\Omega_n + \delta z) = \frac{1}{e^{i2\pi n + \delta z/T} - 1} = \frac{1}{e^{\delta z/T}-1}$$

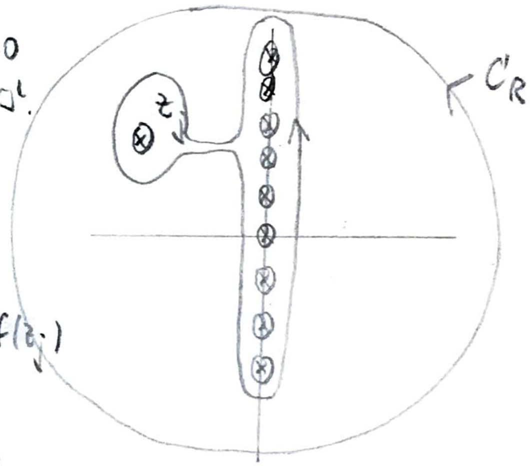
$$= \frac{1}{\delta z/T + O(|\delta z|^2)} = \frac{T}{\delta z} + O(1)$$

1

$\rightsquigarrow$ $n(z)$ has simple poles at $z = i\Omega_n$ with residue T

Consider

$$I = \lim_{R\to\infty} \oint_{C_R} \frac{dz}{2\pi i}\, n(z) f(z) \underset{\substack{f(z\to\infty)\to 0 \\ \text{fast enough}}}{=} 0$$



1

$$= T\sum_n f(i\Omega_n) + \sum_j n(z_j)\,\mathrm{Res}\, f(z_j)$$

$$\rightsquigarrow \underline{\underline{S = \sum_j n(z_j)\,\mathrm{Res}\, f(z_j)}}$$

1