PHYS 610 Mathematical Methods for Scientists F 2024

Problem Assignment # 9 11/27/2024
due 12/04/2024

II.4.1. 1-d Fourier transforms

Consider a function f of one real variable x. Calculate the Fourier transforms $\hat{f}(k) = \int dx\, e^{-ikx}\, f(x)$ of the following functions:

a) $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

b) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

c) $f(x) = e^{-(x/x_0)^2}$.

(3 points)

II.4.2. 3-d Fourier transforms

Consider a function f of one vector variable $\boldsymbol{x} \in I\!R^3$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x}) \quad .$$

Calculate the Fourier transforms of the following functions:

a) $f(\boldsymbol{x}) = \begin{cases} 1 & \text{for } r < r_0 \qquad (r = |\boldsymbol{x}|) \\ 0 & \text{otherwise} \end{cases}$.

b) $f(\boldsymbol{x}) = 1/r$.

hint: Consider $g(\boldsymbol{x}) = \frac{1}{r}\, e^{-r/r_0}$ and let $r_0 \to \infty$.

c) $f(\boldsymbol{x}) = e^{-\alpha \boldsymbol{x}^2}$ with $\alpha \in I\!R$, $\alpha > 0$.

(3 points)

II.4.3. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\; e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm \int dt\, e^{izt}\, f_\pm(t) \qquad (\pm \text{ for } \mathrm{sgn}(\mathrm{Im}\, z) = \pm 1)$$

with z a complex frequency and $f_\pm(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) = \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon\to 0} \epsilon/(x^2+\epsilon^2) = \pi\,\delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in ch. II §4.5.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

II.4.5. Generalized functions derived from generalized functions

Prove Proposition 7 in ch.II §4.4, which says

Let $f(x)$ and $g(x)$ be generalized functions defined by sequences $f_n(x)$ and $g_n(x)$. Then

a) the sum $f(x) + g(x)$ defined by the sequence $f_n(x) + g_n(x)$, and

b) the derivative $f'(x)$ defined by the sequence $f'_n(x)$, and

c) $h(x) = f(ax+b)$ defined by the sequence $f_n(ax+b)$, and

d) $\varphi(x)\,f(x)$ defined by the sequence $\varphi(x)\,f_n(x)$ with φ a fairly good function,and

e) $\hat{f}(k)$ defined by the sequence $\hat{f}_n(k) = FT(f_n)(k)$.

are all generalized functions.

(7 points)

2.4.1. **1-d Fourier transforms**

Consider a function f of one real variable x. Calculate the Fourier transforms $\hat{f}(k) = \int dx\, e^{-ikx}\, f(x)$ of the following functions:

a) $f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

b) $f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$.

c) $f(x) = e^{-(x/x_0)^2}$.

(3 points)

Solution

2.4.1 a) $\hat{f}(\lambda) = \int dx\, e^{-i\lambda x}\, \Theta(|x| \le 1) = \int_{-1}^{1} dx\, e^{-i\lambda x} = \frac{-1}{i\lambda}\left(e^{-i\lambda x} - e^{i\lambda x}\right)$

$= \frac{1}{i\lambda}\, 2i \sin\lambda = \frac{2}{\lambda}\sin\lambda$

b) $\hat{f}(\lambda) = \int_{-1}^{1} dx\, e^{-i\lambda x}(1-|x|) = \int_{-1}^{0} dx\, e^{-i\lambda x}(1+x) + \int_{0}^{1} dx\, e^{-i\lambda x}(1-x)$

$= \frac{2}{\lambda}\sin\lambda + \frac{e^{-i\lambda x}}{-i\lambda}\left(x - \frac{1}{-i\lambda}\right)\Big|_{-1}^{0} - \frac{e^{-i\lambda x}}{-i\lambda}\left(x - \frac{1}{-i\lambda}\right)\Big|_{0}^{1}$

$= \frac{2}{\lambda}\sin\lambda - \frac{1}{(i\lambda)^2} + \frac{1}{i\lambda}e^{i\lambda}\left(-1+\frac{1}{i\lambda}\right) + \frac{1}{i\lambda}e^{-i\lambda}\left(1+\frac{1}{i\lambda}\right) - \frac{1}{(i\lambda)^2}$

$= \frac{2}{\lambda}\sin\lambda + \frac{2}{\lambda^2} - \frac{1}{i\lambda}2i\sin\lambda + \frac{1}{(i\lambda)^2}2\cos\lambda$

$= \frac{2}{\lambda^2}(1-\cos\lambda)$

c) $\hat{f}(\lambda) = \int dx\, e^{-i\lambda x}\, e^{-(x/x_0)^2} = x_0 \int dx\, e^{-x^2 - i\lambda x_0 x}$

$= x_0 \int dx\, e^{-(x^2 + 2i(\lambda x_0/2)x - \lambda^2 x_0^2/4)}\, e^{-\lambda^2 x_0^2/4}$

$= x_0\, e^{-\lambda^2 x_0^2/4} \int dx\, e^{-(x + i\lambda x_0/2)^2}$

$= x_0\, e^{-\lambda^2 x_0^2/4} \int dx\, e^{-x^2}$

$= \sqrt{\pi}\, x_0\, e^{-\lambda^2 x_0^2/4}$

II.4.2. **3-d Fourier transforms**

Consider a function f of one vector variable $\boldsymbol{x} \in \mathbb{R}^3$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(\boldsymbol{k}) = \int d\boldsymbol{x}\ e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, f(\boldsymbol{x}) \quad .$$

Calculate the Fourier transforms of the following functions:

a) $f(\boldsymbol{x}) = \begin{cases} 1 & \text{for } r < r_0 \qquad (r = |\boldsymbol{x}|) \\ 0 & \text{otherwise} \quad . \end{cases}$

b) $f(\boldsymbol{x}) = 1/r \quad .$

hint: Consider $g(\boldsymbol{x}) = \frac{1}{r}\, e^{-r/r_0}$ and let $r_0 \to \infty$.

c) $f(\boldsymbol{x}) = e^{-\alpha \boldsymbol{x}^2}$ with $\alpha \in \mathbb{R}$, $\alpha > 0$.

(3 points)

Solution

2.4.2 a) $f(\vec{x}) = f(r)$

$\rightsquigarrow$ w.l.g. choose $\vec{k} = (0,0,k)$

$$\rightsquigarrow \underline{\hat{f}(\vec{k})} = \hat{f}(k) = \int d\vec{x}\, e^{-ikz}\, \Theta(r_0 - r) = \int_0^{r_0} dr\, r^2 \int_{-1}^{1} d\cos\vartheta \int_0^{2\pi} d\varphi\, e^{-ikr\cos\vartheta}$$

$$= 2\pi \int_0^{r_0} dr\, r^2 \frac{1}{-ikr}\left[e^{-ikr} - e^{ikr}\right]$$

$$= 2\pi \int_0^{r_0} dr\, r \frac{1}{k}\, 2\sin kr = \frac{4\pi}{k} \int_0^{r_0} dr\, r \sin kr$$

$$= \frac{4\pi}{k^3} \int_0^{kr_0} dx\, x \sin x = \frac{4\pi}{k^3}\left[\sin x - x\cos x\right]_0^{kr_0}$$

(1) $$= \underline{\underline{\frac{4\pi}{k^3}\left[\sin kr_0 - kr_0 \cos kr_0\right]}}$$

b) $$\underline{\hat{g}(\vec{k})} = \hat{g}(k) = \int d\vec{x}\, e^{-ikr\cos\vartheta}\, \frac{1}{r} e^{-r/r_0} = 2\pi \int_0^{\infty} dr\, r^2 \int_{-1}^{1} d\cos\vartheta\, e^{-ikr\cos\vartheta} \times \frac{1}{r} e^{-r/r_0}$$

$$= 2\pi \int_0^{\infty} dr\, r\, e^{-r/r_0} \frac{1}{ikr}\, 2i \sin kr$$

$$= \frac{4\pi}{k} \int_0^{\infty} dr\, \sin kr\, e^{-r/r_0} = \frac{4\pi r_0}{k} \int_0^{\infty} dx\, e^{-x} \sin(x(kr_0))$$

$$= \frac{4\pi r_0}{k} \frac{-1}{1+(kr_0)^2}\left[\sin kr_0 x + kr_0 \cos kr_0 x\right] e^{-x} \Big|_0^{\infty}$$

(1) $$= \frac{4\pi r_0}{k} \frac{1}{1+(kr_0)^2}\, kr_0 = \underline{\frac{4\pi r_0^2}{1+k^2 r_0^2}}$$

$$\lim_{r_0\to\infty} g(\vec{x}) = f(\vec{x}) \quad \rightsquigarrow \quad \lim_{r_0\to\infty} \hat{g}(\vec{k}) = \hat{f}(\vec{k})$$

(1) $$\rightsquigarrow \underline{\underline{\hat{f}(\vec{k}) = \hat{f}(k) = \frac{4\pi}{k^2}}}$$

c) $\hat{f}(\vec{\lambda}) = \int d\vec{x}\, e^{-i\vec{\lambda}\cdot\vec{x}}\, e^{-\alpha\vec{x}^2} = \prod_{i=1}^{3} \int dx_i\, e^{-i\lambda_i x_i}\, e^{-\alpha x_i^2}$

Problem 2.4.1 with $1/x_0^2 = \alpha$

$= \prod_{i=1}^{3} \sqrt{\frac{\pi}{\alpha}}\, e^{-\lambda_i^2/4\alpha} = \left(\frac{\pi}{\alpha}\right)^{3/2} e^{-\vec{\lambda}^2/4\alpha}$

2.4.3. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\ e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm i \int dt\, e^{izt}\, f_{\pm}(t) \qquad (\pm \text{ for } \operatorname{sgn}(\operatorname{Im} z) = \pm 1)$$

with z a complex frequency and $f_{\pm}(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) = \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon \to 0} \epsilon/(x^2 + \epsilon^2) = \pi\, \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

Solution

II.4.3.) We have (← assume convergence)

$$F(\omega \pm i0) = \pm \int dt\, \Theta(\pm t)\, e^{i(\omega \pm i0)t} f(t) = \pm \hat{f}_\pm(\omega)$$

a) $$\underline{\hat{f}_+(\omega)} = \int_0^\infty dt\, e^{i\omega t} e^{-t/\tau} = \frac{-1}{i\omega - 1/\tau} = \underline{\frac{\tau}{1 - i\omega\tau}}$$

$$\underline{\hat{f}_-(\omega)} = \int_{-\infty}^0 dt\, e^{i\omega t} e^{t/\tau} = \frac{1}{i\omega + 1/\tau} = \underline{\frac{\tau}{1 + i\omega\tau}}$$

(1) $$\rightsquigarrow \underline{F''(\omega)} = \frac{1}{2i}\left(\hat{f}_+(\omega) + \hat{f}_-(\omega)\right) = \frac{1}{2i}\left(\frac{\tau}{1 - i\omega\tau} + \frac{\tau}{1 + i\omega\tau}\right) = \frac{2\tau}{2i}\frac{1}{1 + (\omega\tau)^2}$$

$$= \underline{\frac{-i\tau}{1 + (\omega\tau)^2}}$$

$$\underline{F'(\omega)} = \frac{1}{2}\left(\hat{f}_+(\omega) - \hat{f}_-(\omega)\right) = \frac{\tau}{2}\left(\frac{1}{1 - i\omega\tau} - \frac{1}{1 + i\omega\tau}\right) = \underline{\frac{i\omega\tau^2}{1 + (\omega\tau)^2}}$$

(1) $$\underline{\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega}} = 0 = \underline{F'(\omega = 0)}$$

b) $$\underline{F(\omega + i0)} = \int_0^\infty dt\, e^{i(\omega + i0)t} e^{i\omega_0 t} = \frac{-1}{i\omega - 0 + i\omega_0} = \underline{\frac{i}{\omega + \omega_0 + i0}}$$

(1) $$\underline{F(\omega - i0)} = -\int_{-\infty}^0 dt\, e^{i(\omega - i0)t} e^{i\omega_0 t} = \frac{-1}{i\omega + 0 + i\omega_0} = \frac{i}{\omega + \omega_0 - i0}$$

$$\rightsquigarrow \underline{F''(\omega)} = \frac{i}{2i}\left(\frac{1}{\omega + \omega_0 + i0} - \frac{1}{\omega + \omega_0 - i0}\right) = \frac{1}{2}\frac{-2i0}{(\omega + \omega_0)^2 + 0^2} = \underline{-i\pi\,\delta(\omega + \omega_0)}$$

$$\underline{F'(\omega)} = \frac{i}{2}\left(\frac{1}{\omega + \omega_0 + i0} + \frac{1}{\omega + \omega_0 - i0}\right) = i\frac{\omega + \omega_0}{(\omega + \omega_0)^2 + 0^2} = \underline{\frac{i}{\omega + \omega_0}}$$

(1) $$\underline{\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega}} = \frac{-i}{-\omega_0} = \frac{i}{\omega_0} = \underline{F'(\omega = 0)}$$

II.4.5. Generalized functions derived from generalized functions

Prove Proposition 1 in ch.2 §4.4, which says

Proposition Let $f(x)$ and $g(x)$ be generalized functions defined by sequences $f_n(x)$ and $g_n(x)$. Then
the following are all generalized functions:

a) $f(x) + g(x)$ defined by the sequence $f_n(x) + g_n(x)$
b) $f'(x)$ defined by $f'_n(x)$
c) $h(x) = f(ax + b)$ defined by the sequence $f_n(ax + b)$
d) $\varphi(x)\,f(x)$ defined by the sequence $\varphi(x)\,f_n(x)$ with φ a fairly good function,
e) $\hat{f}(k)$ defined by the sequence $\hat{f}_n(k) = FT(f_n)(k)$.

(7 points)

Solution

For each of the statements in the proposition we must show
(i) the sequence in question is a sequence of good functions,
(ii) the sequence in question is a regular sequence,
(iii) different choices of equivalent sequences f_n, g_n lead to equivalent sequences that define the new functions.

Property (i) is true for all statements by §4.3 remark (2). 1pt

Now check (ii) and (iii) for the various statements:
a) $\lim_{n\to\infty} \int dx\,(f_n(x) + g_n(x))\,F(x) = \lim_{n\to\infty} \int dx\,f_n(x)\,F(x) + \lim_{n\to\infty} \int dx\,g_n(x)\,F(x)$
The limits on the rhs exist since f_n and g_n are regular sequences, and hence the limit on the lhs exists, so (ii) is true. 1pt

Also, different equivalent sequences f_n and g_n lead to the same limiting values on the rhs $\Rightarrow$ the resulting sequences $f_n + g_n$ are all equivalent, so (iii) is true. 1pt

b) $\lim_{n\to\infty} \int dx\,f'_n(x)\,F(x) = -\lim_{n\to\infty} \int dx\,f_n(x)\,F'(x)$
But $F' \in \gamma \Rightarrow$ the limit on the rhs exists and is the same for all equivalent sequences $f_n(x)$.
$\Rightarrow$ The sequences $f'_n(x)$ on the lhs are regular and equivalent of the sequences f_n are equivalent.
$\Rightarrow$ (ii) and (ii) are true for Statement b). 1pt

The same arguments apply to
c) $\lim{}_n \to \infty \int dx\,f_n(ax + b)\,F(x) = \frac{1}{|a|} \lim_{n\to\infty} \int dx\,f_n(x)\,F((x - b)/2)$ 1pt
and
d) $\lim_{n\to\infty} \int dx\;(\varphi(x)\,f_n(x))\;F(x) = \lim_{n\to\infty} \int dx\,f_n(x)\,(\varphi(x)\,F(x))$ 1pt
since $F((x - b)/a)$ and $\varphi(x)\,F(x)$ are both good functions.
Finally,
e) $\lim_{n\to\infty} \int dk\,hatf_n(k)\,\hat{F}(k) = \lim_{n\to\infty} \int dx\,f_n(x)\,F(-x)$
by Parseval's theorem, and hence the same arguments apply again. 1pt