Chapter 0 | Relativistic Mechanics

§1 The axioms of Mechanics

1.1 One free point particle

Consider a point particle whose state at time t is completely determined by its position $\vec{x}$ and velocity $\vec{v} = d\vec{x}/dt \equiv \dot{\vec{x}}$ at that time.

remark: (1) $\vec{x}$ is a vector in a 3-dim. Euclidean space

question: Given $\vec{x}$ and $\vec{v}$ at an initial time t_0, what determines $\vec{x}$ and $\vec{v}$ at later times?

axiom 1: The motion of the particle is completely determined by a function $L(\vec{x}(t), \vec{v}(t), t) \in \mathbb{R}$ called the Lagrangian. The physical paths $\vec{x}(t)$ are those that minimize the action

$$\boxed{S = \int_{t_-}^{t_+} dt\, L(\vec{x}, \vec{v}, t)}$$

remark: (2) This axiom is called the principle of least action, or Hamilton's principle.

axiom 2: There exist certain coordinate systems, and time scales, in that the Lagrangian of a free point particle, i.e., one whose motion is independent of anything else in the universe, is a fct. of $\vec{v}^2$ only:

$$\boxed{L_0(\vec{x},\vec{v},t) = L_0(\vec{v}^2)}$$

remark: (3) Plausibility arguments: Empty space is

(i) homogeneous $\rightsquigarrow L_0 = L_0(\cancel{\vec{x}},\vec{v},t)$

(ii) isotropic $\rightsquigarrow L_0 = L_0(\vec{v}^2,t)$

and time in an empty universe is homogeneous $\rightsquigarrow$

(iii) $L_0 = L_0(\vec{v}^2,\cancel{t})$ (is called an inertial system (∃!)

def. 1: Any such coordinate system, plus the corresponding time scale

def. 1. $\boxed{m(\vec{v}^2) := 2\, dL_0/d\vec{v}^2}$ is called the mass of the particle

axiom 3: The mass is positive definite, $\boxed{m(\vec{v}^2) > 0}$

def. 2. The particle's momentum is defined by

$$\boxed{\vec{p} := \partial L_0/\partial\vec{v}}$$

remark: (4) This defines the momentum whether or not the particle is free. The quantity

$$\tilde{\pi} := \partial L/\partial\vec{v}$$

is sometimes called generalized momentum

axiom 4: (Galileo) The mass of a free point particle is a constant independent of $\vec{v}^2$. $\boxed{m_G(\vec{v}^2) \equiv m = \text{const.}}$

axiom 4': (Einstein) The mass of a free particle has the form $\boxed{m_E(\vec{v}^2) = m/\sqrt{1-\vec{v}^2/c^2}}$ with m the Galilean mass and remark: (5) For the Lagrangian this implies

$$\boxed{L_0^G(\vec{v}) = \frac{m}{2}\vec{v}^2}$$

$$\boxed{L_0^E(\vec{v}) = -mc^2\sqrt{1-\vec{v}^2/c^2}}$$

(6) For $v \ll c$, $L_0^E(\vec{v}) = -mc^2 + \frac{m}{2}\vec{v}^2\left[1+O(v^2/c^2)\right]$

$\rightarrow$ Galilean mechanics is a limiting case of Einstein's mechanics (NB: The Lagrangian is unique only up to a constant)

(7) In the context of Einstein's mechanics, aka Special Relativity, m is called <u>rest mass</u>, and $E_0 := mc^2$ is called <u>rest energy</u>

1.2 Potentials

Q: What about particles that are not free, but interact with their environment?

axiom 5: The effect of the environment is described by

(a) a scalar potential $U(\vec{x},t)$, and

(b) a vector potential $\vec{V}(\vec{x},t)$

such that the Lagrangian is

$$\boxed{L(\vec{x},\vec{v},t) = L_0(\vec{v}) - U(\vec{x},t) + \vec{v}\cdot\vec{V}(\vec{x},t)}$$

remark: (1) U and $\vec{V}$ are determined either by experiment, or by another theory, not by mechanics.

example: (1) Particle in a gravitational field. U is given by experiment (Kepler, $U = -GMm/|\vec{x}|$), or by GR (Einstein), $\vec{V} = 0$ NB: Within GR, U is v-dependent, see the added remark on p. 0-14

(2) Charged particle in a electromagnetic field. U and $\vec{V}$ determined by Maxwell's Eqns (see el I):

remark: (2) Part of our goal this term is to figure out what U and $\vec{V}$ are in this case.

§2 The Euler-Lagrange equations

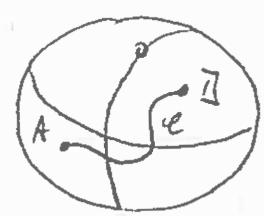
2.1 Three classic problems

(a) The brachistochrone problem (John Bernoulli 1696)

A massive particle moves from point A to point B under the force of gravity along a path $\mathcal{C}$. Which $\mathcal{C}$ results in the shortest passage time?

(b) The geodesic problem (John Bernoulli 1697)

Two points A and B on a 2-sphere S_2 (or any manifold) are connected by a curve $\mathcal{C} \subset S_2$. Which $\mathcal{C}$ is the geodesic, i.e., has the shortest length?



(c) The isoperimetric problem, aka Dido's problem (Pappus of Alexandria, 3rd century CE, Jacob Bernoulli 1690s)

Consider a closed curve $\mathcal{C} \subset \mathbb{R}^2$ with fixed length ℓ. Which shape of $\mathcal{C}$ encloses the largest area?

remark: (1) All three ask for the extremum of a functional under the variation of a function.

(2) The isoperimetric problem involves a constraint.

(3) Calculus does not provide the answer: it can only find extrema of functions under the variation of its argument. $\rightarrow$ Need new formalism ("calculus of variations", Euler, Lagrange)

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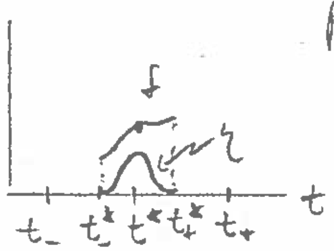
2.2 The fundamental lemma of the calculus of variations

lemma: Let $\mathcal{J} = [t_-, t_+] \subset \mathbb{R}$ and $f: \mathcal{J} \to \mathbb{R}$ a continuous fct. If

$$\boxed{\int_{t_-}^{t_+} dt\, \eta(t) f(t) = 0} \qquad (*)$$

for every fct η that is cont. diff'able on $\mathcal{J}$ and obeys $\eta(t_\mp) = 0$

then $\boxed{f(t) = 0 \ \ \forall t \in \mathcal{J}}$ $(**)$



proof: Suppose $(**)$ does not hold. $\rightarrow \exists\, t^* \in \mathcal{J}: f(t^*) \neq 0$

Continuity $\rightarrow \exists\, [t_-^*, t_+^*] = \mathcal{J}^* \subset \mathcal{J}$ with $t^* \in \mathcal{J}^*$ such that $f(t) \neq 0 \ \forall t \in \mathcal{J}^*$, and > 0 wlog.

Define $\eta(t) = \begin{cases} (t-t_-^*)^2 (t_+^* - t)^2 & \text{for } t \in \mathcal{J}^* \\ 0 & \ldots \end{cases}$

remark: (1) The logical structure of this proof is as follows:

The lemma asserts

$$(*) \text{ true } \forall \eta \Rightarrow (**) \text{ true} \qquad (+)$$

Since $A \Rightarrow B$ is equivalent to not $B \Rightarrow$ not A, $(+)$ is equivalent to

$$(**) \text{ not true} \Rightarrow (*) \text{ not true } \forall \eta$$
$$\Rightarrow \exists \eta : (*) \text{ is not true.} \qquad (++)$$

Therefore, $(+)$ (i.e., the lemma) is equivalent to

$(**)$ not true $\Rightarrow$ we can find an η such that $(*)$ is not true.

We assumed the former and constructed an η that makes $(*)$ false. This proves $(++)$, which by basic logic is equivalent to $(+)$.

Then η and η' are both differentiable on I and $\eta(t_\mp)\cdot\eta'(t_\mp)=0$ and $\eta(t)>0 \;\forall\; t_-^* < t < t_+^*$.

$$\rightsquigarrow \int_{t_-}^{t_+} dt\, \eta(t) f(t) = \int_{t_-^*}^{t_+^*} dt\, \underbrace{\eta(t) f(t)}_{>0} > 0$$

This contradicts $(*)$ $\rightsquigarrow$ $(**)$ must be true $\square$

2.3 The Euler-Lagrange equations

Generalize §1 to a mechanical system with f degrees of freedom, characterized by f generalized positions

$$q(t) = \{q_1(t), \dots, q_f(t)\}$$

and f generalized velocities

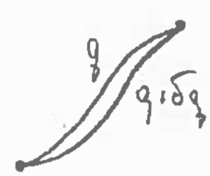
$$\dot{q}(t) = \{\dot{q}_1(t), \dots, \dot{q}_f(t)\}$$

Let the system be conservative, i.e., L has no explicit t-dependence

Lagrangian: $L(q(t), \dot{q}(t)) = L(q_1(t), \dots, q_f(t); \dot{q}_1(t), \dots, \dot{q}_f(t))$

action: $$S = \int_{t_-}^{t_+} dt\, L(q(t), \dot{q}(t))$$

Consider variations of the path, $q(t) \to q(t) + \delta q(t)$ that are smooth and keep the starting and end points fixed. Let the resulting variation of S be δS. Then the extremals of S are given by the requirement that $\delta S = 0$ to linear order in δq:



extremals: $0 \stackrel{!}{=} \delta S = \int dt \left[\sum_{i=1}^{f} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{f} \frac{\partial L}{\partial \dot q_i} \delta \dot q_i \right]$

$$\stackrel{\text{part. int}}{=} \int dt \sum_{i=1}^{f} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot q_i} \right) \delta q_i \quad \forall \, \delta q_i(t)$$

remark: (0) This is the quick and dirty (like LL) way of doing it. For a more careful proof see, e.g., Elsgolc.

$\to (\dots) = 0$ by the fundamental lemma

theorem: Physical paths obey the Euler-Lagrange eqs

$$(*) \quad \boxed{\frac{d}{dt} \frac{\partial L}{\partial \dot q_i} = \frac{\partial L}{\partial q_i}} \quad (i = 1, \dots, f)$$

remark: (1) EL eqs are f coupled ODEs for f fcts $q_i(t)$.

(2) $(*)$ is necessary for S to be minimal, but not sufficient.

(2') This works regardless of whether we consider a mechanical system, where t is the physical time, or some other extremization problem (such as the geodesic problem) where the parameter t that parametrizes the path has nothing to do with time.

(3) Some general consequences of $(*)$:

(i) If L is independent of q_i ("cyclic variable") then $\pi_i(q, \dot q) = \frac{\partial L}{\partial \dot q_i}(q, \dot q)$ is constant along any physical path. π_i is called the "momentum conjugate to q_i". (This is obvious from $(*)$.) If $V = 0$, then $\vec\pi = \vec p$.

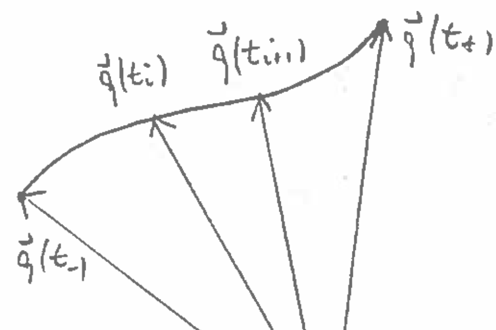
(ii) $H(q, \dot q) := \sum_{i=1}^{f} \dot q_i p_i - L(q, \dot q)$ is constant along any physical path and called energy (or Jacobi's integral more generally). This is less obvious, see PH 4 S 611 for why it's true.

example: Example (2) from p 0-8 f.p.

Problem 0.2.1

Brachistochrone

p 0-8 f.p.



(2) Suppose a particle moves in R_2 on a path $\mathcal{C}$ with parametrization $\vec{q}(t)$. Let the speed of the particle be $v(\vec{q})$.

Time to go from $\vec{q}(t_i)$ to $\vec{q}(t_{i+1} = t_i + \delta t)$: $T_i = \frac{1}{v(\vec{q}(t_i))}\sqrt{(\vec{q}(t_{i+1}) - \vec{q}(t_i)}$

→ Time to go from $\vec{q}(t_-)$ to $\vec{q}(t_+)$:

$$\underline{T(\mathcal{C})} = \lim_{N\to\infty} \sum_{i=1}^{N-1} \frac{1}{v(\vec{q}(t_i))}\sqrt{(\vec{q}(t_{i+1}) - \vec{q}(t_i))^2} = \underline{\int_{t_-}^{t_+} dt \, \frac{1}{v(\vec{q}(t))}\sqrt{(\dot{\vec{q}}(t))^2}}$$

→ Lagrangian $\underline{L(\vec{q}, \dot{\vec{q}}, \cancel{t})} = |\dot{\vec{q}}| / v(\vec{q})$

2.4 Variational problems with a constraint

def.: The functional $S'_1 = \int_{t_-}^{t_+} dt\, L_1(q,\dot q,t)$

is called stationary with constraint

$$\boxed{S'_2 = \int_{t_-}^{t_+} dt\, L_2(q,\dot q,t) = \text{const}} \quad (*)$$

if $\delta S'_1 = 0$ for all variations δq of the path that obey $(*)$

theorem: If a path extremizes S'_1 and does not also extremize S'_2, then there exists a constant λ such that the path extremizes

$$\boxed{S'_3 = \int_{t_-}^{t_+} dt\, \left[L_1(q,\dot q,t) + \lambda L_2(q,\dot q,t)\right]}$$

and λ is determined by $(*)$. proof: Books on calculus of variations (e.g., Elsgolc)

remark: (1) λ is called Lagrange multiplier

example: (1) Let $q(t) = (x(t), y(t))$ be a closed path in $\mathbb{R}_2$.

Then the area enclosed by the path is

$$A = \frac{1}{2}\oint dt\, \left[\dot x(t) y(t) - \dot y(t) x(t)\right]$$

and the length of the path is

$$l = \oint dt \sqrt{\dot x^2(t) + \dot y^2(t)}$$

Dido's problem is to maximize A under the constraint $l = \text{const}$. → We need to consider

$$\boxed{L = \frac{1}{2}(x\dot y - y\dot x) + \frac{1}{2}\lambda\sqrt{\dot x^2 + \dot y^2}}$$

Problem 0.2.2
Dido's problem

Problem 0.2.3
Geodesics on S_2

Problem Week 1 (0.2.1–3)

(see 610 ch 1 § 5)

2.5 Euler-Lagrange eqs for fields

Consider a Lagrangian L that depends on a scalar field $\phi(\vec{x},t)$ and its time and spatial derivatives $\partial^\mu \phi(\vec{x},t)$.

remark: (1) $\partial^0 = \frac{1}{c}\partial_t$, $(\partial^1,\partial^2,\partial^3) \equiv \vec{\partial} = -\vec{\nabla}_{\vec{x}}$, see PHYS 610

Discretize real space => the field $\phi(x) = \phi(\vec{x},t)$ can be considered a system with f degrees of freedom in the limit $f \to \infty$ if we identify $\phi(\vec{x}_1,t) \equiv q_1(t)$, $\phi(\vec{x}_2,t) \equiv q_2(t)$, etc.

The Lagrangian now becomes a

Lagrangian density: $\mathcal{L}(\phi(\vec{x},t), \partial^\mu\phi(\vec{x},t))$ that depends on spatial gradients in addition to time derivatives, and the

Lagrangian: $L = \int d\vec{x}\, \mathcal{L}(\phi(\vec{x},t), \partial^\mu\phi(\vec{x},t))$ is the spatial integral over $\mathcal{L}$

The

action: $S = c\int dt\, L = \int dx^0 \int d\vec{x}\, \mathcal{L}(\phi(x), \partial^\mu\phi(x)) = \int d^4x\, \mathcal{L}(\phi,\partial^\mu\phi)$

is defined as for $f < \infty$

extremalize: $0 \overset{!}{=} \delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right]$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi \quad \forall\,\delta\phi$$

2.1 ->

Problem 0.2.4

Functional derivative

$$\Rightarrow \boxed{\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{\partial\mathcal{L}}{\partial\phi}} \quad (*)$$

Remark: (1) Here we use the summation convention, i.e., $\sum_{\mu=0}^{3}$ over repeated indices μ is implied

(2) PHYS 610 $\leadsto$ $\partial_\mu = \frac{\partial}{\partial x^\mu}$ transforms as a covariant tensor. For the same reason, $\frac{\partial}{\partial(\partial_\mu \phi)}$ transforms as a contravariant tensor, so $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ really is a proper contraction!

(3) The significance of covariant vs contravariant quantities depends on whether we define our field ϕ on a Euclidean space or a Minkowski space (we still have that freedom)

(4) (*) is the EL eq. for a scalar field $\phi(x)$. Generalization to tensor fields is straightforward: just add (discrete) indices for the components.

(5) (*) is a PDE, as opposed to the corresponding ODE in mechanics!

(6) There is no fundamental reason why $\mathcal{L}$ can't depend on higher derivatives. In Maxwell theory it does not, so we restrict ourselves to first derivatives.

example: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \frac{m^2}{2}(\phi(x))^2$ "massive scalar field"

$$\rightarrow \underline{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}} = \frac{\partial}{\partial(\partial_\mu \phi)} \frac{1}{2}(\partial_\nu \phi)(\partial_\lambda \phi) g^{\lambda\nu}$$

$$= \frac{1}{2}\delta^\mu{}_\nu (\partial_\lambda \phi) g^{\lambda\nu} + \frac{1}{2}(\partial_\nu \phi)\delta^\mu{}_\lambda g^{\lambda\nu}$$

$$= \frac{1}{2}(\partial_\lambda \phi) g^{\lambda\mu} + \frac{1}{2}(\partial_\nu \phi) g^{\mu\nu}$$

$$= \frac{1}{2}\partial^\mu \phi + \frac{1}{2}\partial^\mu \phi = \underline{\partial^\mu \phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$\rightarrow$ The EL eq is $\underline{\underline{(\partial_\mu \partial^\mu + m^2)\phi(x) = 0}}$

"Klein-Gordon eq"

h.w. $\rightarrow$ Problem 0.2.5
Massive scalar field

remark: (7) The Lagrange multiplier method from §2.4 is very useful also in field theory, both classical (e.g., MSR formalism for hydrodynamics) and quantum (e.g., NLσMs for electrons), but we will not use it in this course.

§3 Relativistic Mechanics

Go back to mechanics for a while

3.1 Newton's first law

lemma: The momentum of a free particle is related to its velocity:

$$\boxed{\vec{p} = m(\vec{v}^2)\vec{v}}$$

proof: $\vec{p} = \frac{\partial L_0}{\partial \vec{v}} = \frac{\partial L_0}{\partial \vec{v}^2} 2\vec{v} = m(\vec{v}^2)\vec{v}$ by §1.1 def. 1

theorem: Newton's 1st law

The physical paths of free particles in an inertial system are straight lines:

$$\boxed{\vec{x}(t) = \vec{x}_0 + \vec{v}t} \quad \text{with } \vec{v} = \dot{\vec{x}} = \text{const.}$$

proof: $\vec{x}$ is cyclic $\Rightarrow$ $\vec{p} = \text{const} \overset{\text{lemma}}{\Rightarrow} \vec{v}\, m(\vec{v}^2) = \text{const}$

$\Rightarrow \vec{v} = \text{const} = \dot{\vec{x}} \Rightarrow \underline{\underline{\vec{x} = x_0 + \vec{v}t}}$ $\square$

remark: (1) Newton's 1st law holds independent of the functional form of $L_0(\vec{v}^2)$!

(2) §1.1 axiom 2 alone implies Newton's 1st law. This shows that the axiom is nontrivial, as there are coordinate systems in which Newton's 1st law does not hold (e.g., a cartesian system fixed to a moving car

0.2 Newton's second law

theorem: Newton's 2^{nd} law

With the Lagrangian given by §1 exercise 1, the eq. of motion takes the form

$$\boxed{\frac{d}{dt}\vec{p}(\vec{x},t) = \vec{F}^{(1)}(\vec{x},t) + \vec{F}^{(2)}(\vec{x},\vec{v},t)}$$

with $\boxed{\vec{F}^{(1)}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \partial_t\vec{V}(\vec{x},t)}$ a velocity-independent force

and $\boxed{\vec{F}^{(2)}(\vec{x},t) = \vec{v}\times(\vec{\nabla}\times\vec{V}(\vec{x},t))}$ a velocity-dependent force

remark: (1) The lhs takes the form mass × acceleration only for a constant mass! → do not memorize this as $\vec{F} = m\vec{a}$!!

proof: consider $\frac{d}{dt}\frac{\partial L}{\partial v^i} = \frac{d}{dt}\Big(\underbrace{\frac{\partial L_0}{\partial v^i}}_{=p_i} + V_i\Big) \overset{EL}{=} \frac{\partial L}{\partial x^i} = -\partial_i U + \partial_i v_j V$

$$\Rightarrow \frac{d}{dt}p_i = \underbrace{-\partial_i U - \partial_t V_i}_{=F^{(1)}_i} - (\partial_j V_i)v^j + v_j\partial_i V^j$$

$$(\vec{v}\times(\vec{\nabla}\times\vec{V}))_i = \varepsilon_{ijl}v^j\varepsilon^{lkm}\partial_k V_m$$
$$= (\delta_i^{\,k}\delta_j^{\,m} - \delta_i^{\,m}\delta_j^{\,k})v^j\partial_k V_m$$
$$= v^j\partial_i V_j - v^j\partial_j V_i = F^{(2)}_i$$

$$= F^{(1)}_i + F^{(2)}_i \quad \square$$

Problem 0.2.6
orbits in homogeneous E and B fields

Problem 0.2.7
harm. osc. coupled to a B-field

example: For a particle with charge e in time-independent electric and magnetic fields $\vec{E}, \vec{B}$ one has (see §1 for a derivation)

$$\vec{F}^{(1)} = e\vec{E}, \quad \vec{F}^{(2)} = \frac{e}{c}\vec{v}\times\vec{B} \quad \text{(Lorentz force)}$$

1.3 Example: Einstein's Law of Falling Bodies

Consider Einsteinian mechanics for a point particle in a linear potential $U(\vec{x},t) = U(\cancel{x},\cancel{y},z,\cancel{t}) = -mgz$ $\quad g \Downarrow \hat{z}$

i.e., $\boxed{L = -mc^2\sqrt{1-\vec{v}^2/c^2} + mgz}$ $\quad \vec{v}^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$

This potential is an approximation that is not consistent with GR. Within GR the force is m(v)g rather than mg, see the homework problem in LL II $88, and remark (1) on p.157-21

step 1: Identify constants of motion

x cyclic → $p_x = \dfrac{\partial L}{\partial v_x} = \dfrac{m\dot{x}}{\sqrt{1-v^2/c^2}}$ = const =: p_x^0

y cyclic → $p_y = \dfrac{\partial L}{\partial v_y} = \dfrac{m\dot{y}}{\sqrt{1-v^2/c^2}}$ = const =: p_y^0

Further, $p_z = \dfrac{\partial L}{\partial v_z} = \dfrac{m\dot{z}}{\sqrt{1-v^2/c^2}}$ but this is not const

→ $\vec{p}^2 = p_x^2+p_y^2+p_z^2 = \dfrac{m^2v^2}{1-v^2/c^2}$

→ $m^2v^2 = p^2(1-v^2/c^2)$ → $v^2(m^2+p^2/c^2) = p^2$

→ $v_i = p_i/\sqrt{m^2+p^2/c^2}$

Consider $\dfrac{d}{dt}(xp_y - yp_x) \overset{p_{x,y}=\text{const}}{=} v_xp_y - v_yp_x = \dfrac{p_xp_y - p_yp_x}{\sqrt{m^2+p^2/c^2}} = 0$

→ $x(t)p_y - y(t)p_x$ = const $\equiv c$ → path lies in a plane that contains the z-axis

Choose coordinate system such that $p_y = 0$ and $c = 0$

→ $y(t) \equiv 0$ path lies within x-z plane

Step 2: Use Newton's 2nd law to find the velocity

$$\frac{d}{dt} p_z = -\frac{d}{dz} U = mg \quad \rightarrow \quad p_z(t) = p_z^0 + mgt$$

$$\rightarrow \quad v_z(t) = \frac{p_z^0/m + gt}{\sqrt{1 + p_x^{0\,2}/m^2c^2 + (p_z^0 + mgt)^2/m^2c^2}}$$

$$v_x(t) = \frac{p_x^0/m}{\sqrt{1 + p_x^{0\,2}/m^2c^2 + (p_z^0 + mgt)^2/m^2c^2}}$$

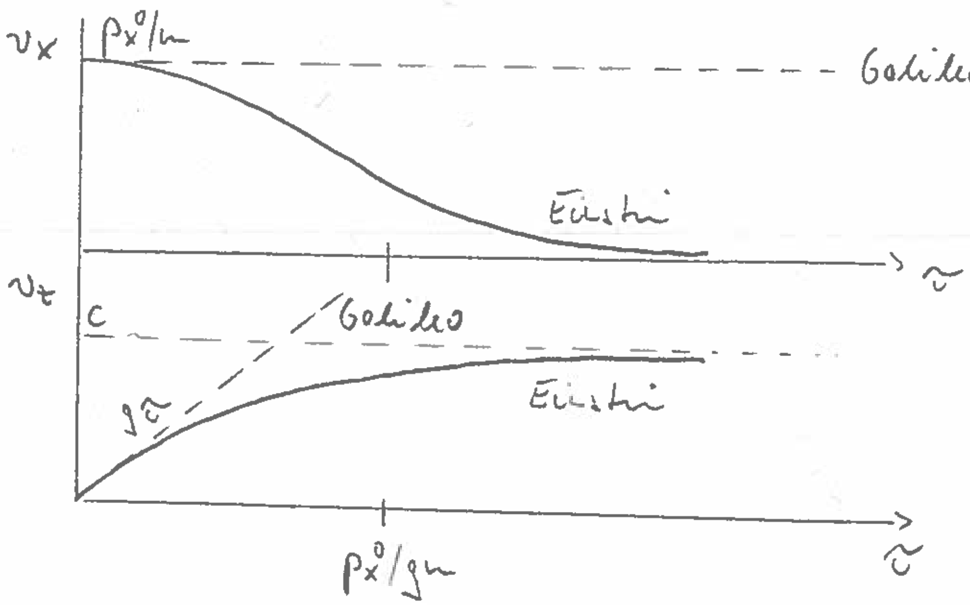
Define $\underline{\tau := t + p_z^0/mg}$

$$\rightsquigarrow \quad \boxed{\begin{pmatrix} v_x(t) \\ v_z(t) \end{pmatrix} = \frac{1}{\left[1 + (p_x^0)^2/m^2c^2 + \tau^2 g^2/c^2\right]^{1/2}} \begin{pmatrix} p_x^0/m \\ \tau g \end{pmatrix}} \qquad (*)$$

Discuss limits:

$\underline{c \rightarrow \infty}$ $\rightsquigarrow$ $\underline{\begin{pmatrix} v_x(t) \\ v_z(t) \end{pmatrix}} \rightarrow \underline{\begin{pmatrix} p_x^0/m \\ g\tau \end{pmatrix}}$ Galilean result ✓

$\underline{t \rightarrow \infty}$ $\rightsquigarrow$ $v_x(t) \rightarrow 0$, $v_z(t) \rightarrow c$ ultrarelativistic limit



step 3: Integrate to find the position. (x) ->

$$\underline{x(t)-x_0} = \frac{p_x^0/m}{g/c}\int_0^{\tau} dt\, \frac{1}{[t^2+\underbrace{\frac{c^2}{g^2}(1+(p_x^0)^2/m^2c^2)}_{=:\,\tau^{*2}}]^{1/2}} = \frac{p_x^0 c}{mg}\,\mathrm{arsh}(\tau/\tau^*)$$

$$\underline{z(t)-z_0} = \frac{g}{g/c}\int_0^{\tau} dt\, \frac{t}{[t^2+\tau^{*2}]^{1/2}} = c\left(\sqrt{\tau^2+\tau^{*2}}-\tau^*\right)$$

where $\tau = t + p_z^0/mg$, $\tau^* = \frac{c}{g}\sqrt{1+(p_x^0)^2/m^2c^2}$

Discuss limits:

$c\to\infty$: $\tau^* \to c/g \to x(t)-x_0 \to \frac{p_x^0 c}{mg}\frac{\tau g}{c} = \frac{p_x^0}{m}\tau$

$z(t)-z_0 \to c\left(\tau^*(1+\frac{1}{2}\tau^2/\tau^{*2}\dots)-\tau^*\right)$

$= \frac{c}{2\tau^*}\tau^2 = \frac{1}{2}g\tau^2$ ← Galilean result ✓

Problem 0.2.8
Relativistic motion in parallel E and B fields.

Problem 0.2.9
Relativistic Coulomb problem

Week 2

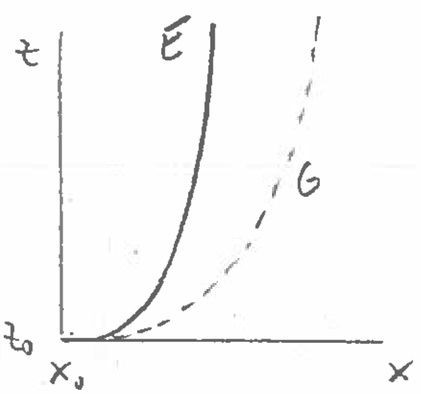
$\tau\to\infty$: $z(t)\to ct$, $x(t) \to \frac{p_x^0 c}{mg}\log(\tau/\tau^*)$

ultrarelativistic limit

step 4: Determine the orbit.

Define $\xi := \frac{mg}{p_x^0 c}(x(t)-x_0) \to \tau/\tau^* = \sinh\xi$

$\to \boxed{z-z_0 = c\tau^*\left[\sqrt{1+\sinh^2\xi}-1\right] = c\tau^*(\cosh\xi - 1)}$ orbit



limits: $c\to\infty$: $z-z_0 \to \frac{1}{g}c^2\frac{1}{2}\xi^2 = \frac{m^2 g}{2(p_x^0)^2}(x-x_0)^2$ parabola ✓

$x\to\infty$: $z-z_0 \propto \cosh\xi$ ultrarelativistic limit