Chapter 1 | Maxwell's equations

§1 The variational principle of classical electrodynamics

1.1 The Maxwellian action

axiom 1: Space and time are described by a 4-dimensional Minkowski space with elements $x^\mu = (ct, \vec{x})$, where t is time, $\vec{x} = (x^1, x^2, x^3)$ is position in real space, and c is a characteristic velocity.

remark: (1) We will choose the metric to be $g_{\mu\nu} = (+,-,-,-)$. We will use Greek indices, $\mu, \nu, \ldots = 0,1,2,3$, to denote coordinates in M^4, and Latin indices, $i, j, \ldots = 1,2,3$, to denote coordinates in the 3-d Euclidian subspace.

axiom 2: Empty space ("vacuum") supports a Minkowski-vector field $A^\mu(x)$ called the electromagnetic 4-vector potential

def. 1: The rank-2 tensor (electromagnetic field $F^{\mu\nu}$) constructed from gradients of A^μ as

$$\boxed{F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)}$$

is called electromagnetic field tensor

remark: (2) $F^{\mu\nu}(x)$ can be represented in terms of a Euclidian vector field and a Euclidian pseudovector field axiom I: (a) The physical field configurations in vacuum are those that minimize the action

$$\boxed{S_{vac} = \frac{-1}{16\pi}\int d^4x \; F_{\mu\nu}(x) F^{\mu\nu}(x)}$$

with $\int d^4x = c\int dt \int d\vec{x}$

(0) The $1/16\pi$ is conventional and depends on the units used

remark: (1) ED is governed by a principle of least action, as is classical mechanics. However, in ED we need to find field configurations $A^\mu(x)$; in mechanics we only need to find paths $\vec{x}(t)$.

(2) $A^\mu(x)$ enters the action only in conjunction with gradients!

(3) $F_{\mu\nu}F^{\mu\nu}$ is a Minkowski scalar so the theory is invariant under Lorentz transfos, but not under Galileo transfos!

(4) Maxwell did not formulate the theory like this. Advantages of this formulation: manifest Lorentz invariance, easy to generalize, analogies to other field theories (GR, particle physics).

(5) Maxwell theory was the first unified field theory.

axiom II: (b) Matter is characterized by (among other things) a 4-vector $J^\mu(x)$ that couples to $A^\mu(x)$ via

$$\boxed{S_{int} = \frac{-1}{c}\int d^4x \; J_\mu(x) A^\mu(x)}$$

J^μ is called 4-current vector

(7) The factor $1/c$ is conventional

The field plus its interaction with a given $J^\mu(x)$ is described by the action $S = S_{vac} + S_{int}$.

remark: (6) This does not include the feedback from the field on the matter. For that, one needs another piece of the action that governs the matter def. 2 : The dual field tensor (unfortunate name) is defined as

$$\tilde{F}^{\mu\nu} := \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} \quad \text{with } \varepsilon^{\mu\nu\kappa\lambda} \text{ the 4-d Levi-Civita tensor}$$

Problem 1 dual field tensor

proposition : $\boxed{\partial_\mu \tilde{F}^{\mu\nu}(x) = 0}$ (*) proof: Problem # 1

remark : (8) The structure of $F^{\mu\nu}$ in terms of gradients of A^μ implies (*).

1.2 Euler - Lagrange equations for fields

Recall classical mechanics :

Lagrangian : $L(q_1(t), q_2(t), \dots, q_f(t); \dot{q}_1(t), \dot{q}_2(t), \dots, \dot{q}_f(t))$

action : $S = \int dt\, L(q(t), \dot{q}(t))$

extremals : $0 \stackrel{!}{=} \delta S = \int dt \left[\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right]$

$\stackrel{\text{part. int.}}{=} \int dt \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right] \delta q_i \quad \forall\, \delta q_i(t)$

$\rightsquigarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$ by the fundamental lemma

Now consider field theory : discretize space $\rightsquigarrow$ a field

$$\phi(x) = \phi(\vec{x}, t)$$

can be considered a system with $f = \infty$ if we identify $\phi(\vec{x}_1, t) \equiv q_1(t)$, $\phi(\vec{x}_2, t) \equiv q_2(t)$, etc. The Lagrangian now becomes a

Lagrangian density : $\mathcal{L}(\phi(\vec{x},t), \partial^\mu \phi(\vec{x},t))$ that depends on spatial gradients in addition to time derivatives, and

Lagrangian : $L = \int d\vec{x}\, \mathcal{L}(\phi(\vec{x},t), \partial^\mu \phi(\vec{x},t))$ is the spatial integral over $\mathcal{L}$.

discussion: Derive the GL eqs, Problem 4, the way we derived (*):

$$S = S[\phi, \phi^*, \vec\nabla\phi, \vec\nabla\phi^*, \vec A, \partial_i A_j]$$

Extremize:

$$0 \overset{!}{=} \delta S = \int d\vec x \left(\frac{\partial \mathcal{L}}{\partial \phi^*}\delta\phi^* + \frac{\partial \mathcal{L}}{\partial \vec\nabla\phi^*}\cdot\delta\vec\nabla\phi^* + c.c. + \frac{\partial \mathcal{L}}{\partial \vec A}\delta\vec A + \frac{\partial \mathcal{L}}{\partial \partial_i A_j}\delta(\partial_i\right.$$

$$= \int d\vec x \left(\frac{\partial \mathcal{L}}{\partial \phi^*} - \vec\nabla\cdot\frac{\partial \mathcal{L}}{\partial \vec\nabla\phi^*} \right)\delta\phi^* + c.c. + \int d\vec x \left(\frac{\partial \mathcal{L}}{\partial A_j} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i A_j)}\right)\delta$$

→ One condition is

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^*} - \vec\nabla\cdot\frac{\partial \mathcal{L}}{\partial \vec\nabla\phi^*} = 0} \qquad (+)$$

Now, $\mathcal{L}_{GL} = r\phi\phi^* + c[(\vec\nabla - ig\vec A)\phi][(\vec\nabla + ig\vec A)\phi^*] + u\phi^2\phi^{*2} +$ terms that don't depend on

$$= r\phi\phi^* + c(\vec\nabla\phi)(\vec\nabla\phi^*) + c(\vec\nabla\phi)ig\vec A\phi^* - cig\vec A\phi\,\vec\nabla\phi^*$$
$$+ cg^2\vec A^2\phi\phi^* + u\phi^2\phi^{*2} + \dots$$

$$\rightsquigarrow \frac{\partial \mathcal{L}}{\partial \phi^*} = r\phi + cig\vec A\vec\nabla\phi + cg^2\vec A^2\phi + 2u\phi^2\phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\vec\nabla\phi^*)} = c\vec\nabla\phi - cig\vec A\phi$$

→ (+) reads explicitly

$$\underline{\underline{0}} = (r + 2u|\phi|^2)\phi - c(\vec\nabla^2\phi - ig\vec\nabla\cdot(\vec A\phi) - ig\vec A\cdot\vec\nabla\phi - g^2\vec A^2\phi)$$

$$= \underline{\underline{(r + 2u|\phi|^2)\phi - c[\vec\nabla - ig\vec A]^2\phi}}$$

which is the same eq. we obtain from $\frac{\delta S}{\delta\phi^*(\vec x)} \overset{!}{=} 0$, see my solution of Problem 4. The other GL eq. follows analogously

The action: $S = c\int dt\, L = \int dx^0 \int d\vec{x}\, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) = \int d^4x\, \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

is defined as for $f < \infty$.

extremal: $$0 \overset{!}{=} \delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi \quad \forall \delta\phi$$

$$\rightarrow \quad \boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi}} \qquad (*)$$

remark: (1) (*) is the Euler-Lagrange equation for the scalar field ϕ. (This is the fast way of deriving it. See Problem 2 for a somewhat more detailed procedure.)

Problem 2 functional derivative

(2) Generalization to more fields is straightforward: just add more (discrete) indices.

Problem 3 Klein-Gordon eq

Problem 4 GL eqs

(3) In general, $\mathcal{L}$ will depend on higher gradients. §1.1 axiom (3) postulates that the Maxwellian $\mathcal{L}$ has zero gradients of $F^{\mu\nu}$, or two gradients of A^μ.

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(4) The EL eqs for fields are PDEs, in contrast to the case $f < \infty$, where they are coupled ODEs!

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1.3 The field equations

§1.1 → the Lagrangian density of Maxwell theory is

$$\boxed{\mathcal{L}(A^\mu(x)) = \frac{-1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{c} A_\mu(x) j^\mu(x)}$$

§1.2 → The Euler-Lagrange equation reads

$$\partial_\lambda \frac{\partial}{\partial(\partial_\lambda A_\alpha(x))} \mathcal{L} = \frac{\partial}{\partial A_\alpha(x)} \mathcal{L}$$

In $\mathcal{L}$, the first term depends only on gradients of $A^\mu(x)$, and the second term is gradient-free.

$$\rightarrow \quad \frac{\partial}{\partial(\partial_\Lambda A_\alpha(x))}\mathcal{L} = \frac{-1}{16\pi}\frac{\partial}{\partial(\partial_\Lambda A_\alpha)} F_{\mu\nu}\, g^{\mu\rho} g^{\nu\lambda} F_{\rho\lambda}$$

$$= \frac{-1}{16\pi}\frac{\partial}{\partial(\partial_\Lambda A_\alpha)}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial_\rho A_\lambda - \partial_\lambda A_\rho\right) g^{\mu\rho} g^{\nu\lambda}$$

$$= \frac{-1}{16\pi} g^{\mu\rho} g^{\nu\lambda}\Big[\left(\delta^\Lambda{}_\mu \delta^\alpha{}_\nu - \delta^\Lambda{}_\nu \delta^\alpha{}_\mu\right)\left(\partial_\rho A_\lambda - \partial_\lambda A_\rho\right) + \left(\delta^\Lambda{}_\rho \delta^\alpha{}_\lambda - \delta^\Lambda{}_\lambda \delta^\alpha{}_\rho\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\Big]$$

$$= \frac{-1}{16\pi}\Big[\left(g^{\Lambda\rho} g^{\alpha\lambda} - g^{\alpha\rho} g^{\Lambda\lambda}\right)\left(\partial_\rho A_\lambda - \partial_\lambda A_\rho\right) + \left(g^{\mu\Lambda} g^{\nu\alpha} - g^{\mu\alpha} g^{\nu\Lambda}\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\Big]$$

$$= \frac{-1}{16\pi}\left[\partial^\Lambda A^\alpha - \partial^\alpha A^\Lambda - \partial^\alpha A^\Lambda + \partial^\Lambda A^\alpha + \partial^\Lambda A^\alpha - \partial^\alpha A^\Lambda - \partial^\alpha A^\Lambda + \partial^\Lambda A^\alpha\right]$$

$$= -\frac{1}{4\pi}\left(\partial^\alpha A^\Lambda - \partial^\Lambda A^\alpha\right) = \frac{1}{4\pi} F^{\alpha\Lambda}$$

and $\displaystyle \frac{\partial}{\partial A_\alpha(x)}\mathcal{L} = \frac{\partial}{\partial A_\alpha}\, \frac{-1}{c} A_\mu j^\mu = -\frac{1}{c}\delta^\alpha{}_\mu j^\mu = -\frac{1}{c} j^\alpha$

$\rightarrow$ the EL-eq. reads $\displaystyle \frac{1}{4\pi}\partial_\Lambda F^{\alpha\Lambda}(x) = -\frac{1}{c} j^\alpha(x)$

or $$\boxed{\partial_\mu F^{\mu\nu}(x) = \frac{4\pi}{c} j^\nu(x)} \quad (*)$$

remark: (1) All physical field configurations must obey $(*)$.

(2) $F^{\mu\nu}$ is defined in terms of A^μ, so $(*)$ should be considered a differential eq. for $A^\mu(x)$. Alternatively we can augment $(*)$ by the proposition from §1.1:

$$\boxed{\partial_\mu \varepsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}(x) = 0} \quad (**)$$

which contains the structure of $F^{\mu\nu}$ in terms of gradients of A^μ. We can then consider $(*) + (**)$ the field eqs for $F^{\mu\nu}(x)$, which is now the fundamental field

§2 Conservation laws, and gauge invariance

2.1 Continuity equation for the 4-current

proposition: The 4-current vector obeys the continuity eq.

$$\boxed{\partial_\mu j^\mu(x) = 0} \quad (*)$$

symm. antisymm.

proof: §1.3 (a) ~> $\partial_\nu j^\nu = \frac{c}{4\pi}\partial_\nu\partial_\mu F^{\mu\nu} = -\frac{c}{4\pi}\partial_\nu\partial_\mu F^{\mu\nu} = 0$

remark: (1) The 4-vector $j^\mu = (c\rho, \vec{j})$ has a time-like component $j^0 = c\rho$ and three space-like components $j^i = j$. ρ is called electric charge density and $\vec{j}$ is called electric current density.

(2) In terms of ρ and $\vec{j}$, $(*)$ reads $\partial_0 c\rho + \partial_i j^i = 0$

But $\partial_0 = \frac{\partial}{\partial x^0} = \frac{1}{c}\partial_t$, and $\partial_i = \frac{\partial}{\partial x^i}$ = i-component of $\vec{\nabla}$

~> $$\boxed{\partial_t \rho(\vec{x},t) + \vec{\nabla}\cdot\vec{j}(\vec{x},t) = 0} \quad (**)$$

is equivalent to $(*)$.

(3) Integrate $(**)$ over a spatial volume V with bounding surface

$$\partial_t \int_V d\vec{x}\, \rho(\vec{x},t) = -\int_V d\vec{x}\, \vec{\nabla}\cdot\vec{j}(\vec{x},t) = -\int_{(V)} d\vec{s}\cdot\vec{j}(\vec{x},t)$$

define $Q := \int_V d\vec{x}\, \rho(\vec{x},t)$ to be the total charge within

~> $\frac{d}{dt}Q = -\int_{(V)} d\vec{s}\cdot\vec{j}(\vec{x},t)$

~> Q can change only via current flowing in or out of the volume V, hence the name "continuity eq."

Week I
Problem 1
(0,1,2,3,4)

2.2 The energy-momentum tensor

def.: The tensor field

$$\boxed{T^{\mu\nu}(x) := \frac{-1}{4\pi} F^{\mu\alpha}(x) F^{\nu}{}_{\alpha}(x) + \frac{1}{16\pi} g^{\mu\nu} F_{\alpha\beta}(x) F^{\alpha\beta}(x)} \qquad (*)$$

is called the electromagnetic energy-momentum tensor.

other T's: $^{\mu\nu}$ from Noether's [illegible]

remark: (1) It is not obvious what this has to do with energy and momentum, see Problem 5 for a hint, and LL for details.

proposition: (1) $T^{\mu\nu}$ is symmetric, $\boxed{T^{\mu\nu}(x) = T^{\nu\mu}(x)}$

(2) $T^{\mu\nu}$ is traceless, $\boxed{T_{\mu}{}^{\mu}(x) = 0}$

proof: (1) $g^{\mu\nu} = g^{\nu\mu}$, and $F^{\mu\alpha}F^{\nu}{}_{\alpha} = g^{\alpha\beta} F^{\mu}{}_{\beta}\, g_{\alpha\gamma} F^{\nu\gamma}$

$$= \delta^{\beta}{}_{\gamma} F^{\mu}{}_{\beta} F^{\nu\gamma} = F^{\nu\alpha} F^{\mu}{}_{\alpha}$$

(2) $-4\pi T^{\mu}{}_{\mu} = F^{\mu\alpha}F_{\mu\alpha} - \frac{1}{4}\underbrace{g^{\mu}{}_{\mu}}_{=4} F_{\alpha\beta}F^{\alpha\beta} = 0$ □

remark: (2) cf. I §3.4 ⇒ $T^{\mu\nu}$ can be decomposed into T^{00} plus a Euclidean vector T^{0i} plus a symmetric Euclidean tensor T^{ij}.

2.3 The continuity equation for the energy-momentum tensor

proposition: In the absence of matter the energy-momentum tensor obeys the continuity eq.

$$\boxed{\partial_{\nu} T_{\mu}{}^{\nu}(x) = 0}$$

proof: §2.2 (*) ⇒ $\partial_{\nu} T_{\mu}{}^{\nu} = \frac{1}{4\pi}\left[-\partial_{\nu} F_{\mu}{}^{\alpha} F^{\nu}{}_{\alpha} + \frac{1}{4}\partial_{\nu}\delta_{\mu}{}^{\nu} F_{\alpha\beta}F^{\alpha\beta}\right]$

$$= \frac{1}{4\pi}\left[-(\partial_\nu F_\mu{}^\alpha)F^\nu{}_\alpha - F_\mu{}^\alpha \partial_\nu F^\nu{}_\alpha + \frac{1}{4}\partial_\mu F_{\alpha\lambda}F^{\alpha\lambda}\right]$$

$\partial_\nu F^\nu{}_\alpha = 0$ by EL

$$F_{\alpha\lambda}\partial_\mu F^{\alpha\lambda} = g_{\alpha\gamma} g_{\lambda\delta} F^{\gamma\delta}\partial_\mu g^{\alpha\varepsilon} g^{\lambda\nu} F_{\varepsilon\nu} = \delta_\gamma{}^\varepsilon \delta_\delta{}^\nu F^{\gamma\delta}\partial_\mu F_{\varepsilon\nu}$$
$$= F^{\varepsilon\delta}\partial_\mu F_{\varepsilon\delta} = (\partial_\mu F_{\alpha\lambda})F^{\alpha\lambda} \to \partial_\mu F_{\alpha\lambda}F^{\alpha\lambda} = 2(\partial_\mu F_{\alpha\lambda})F^{\alpha\lambda}$$

$$= \frac{1}{4\pi}\left[-(\partial_\nu F_\mu{}^\alpha)F^\nu{}_\alpha + \frac{1}{2}(\partial_\mu F_{\alpha\lambda})F^{\alpha\lambda}\right]$$

Problem 1 $\to$ $0 = \partial_\mu F_{\alpha\lambda} + \partial_\alpha F_{\lambda\mu} + \partial_\lambda F_{\mu\alpha}$

$$= \frac{1}{4\pi}\left[-(\partial_\nu F_\mu{}^\alpha)F^\nu{}_\alpha - \frac{1}{2}(\partial_\alpha F_{\lambda\mu})F^{\alpha\lambda} - \frac{1}{2}(\partial_\lambda F_{\mu\alpha})F^{\alpha\lambda}\right]$$
$$= \frac{1}{4\pi}\left[-(\partial_\nu F_{\mu\alpha})F^{\nu\alpha} + \frac{1}{2}(\partial_\alpha F_{\mu\lambda})F^{\alpha\lambda} + \frac{1}{2}(\partial_\lambda F_{\mu\alpha})F^{\lambda\alpha}\right] = \underline{0}$$

corollary: In the presence of matter the continuity eq. gets modified to

$$\boxed{\partial_\nu T_\mu{}^\nu(x) = -\frac{1}{c}F_\mu{}^\nu(x)\, j_\nu(x)}$$

Problem 6)
proof of corollary

proof: Problem 6

Problem 7
EM tensor for the Klein-Gordon field

remark: For any rank-$(n+1)$ tensor field $t^{\mu\alpha_1\ldots\alpha_n}(x)$ the continuity eq. $\partial_\mu t^{\mu\alpha_1\ldots\alpha_n}(x) = 0$ implies a conservation law for the rank-n tensor $t^{0\alpha_1\ldots\alpha_n}(x)$ by the arguments from § 2.1. (a) in § 2.1 is the case $n=0$; the proposition above is the case $n=1$.

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2.4 Gauge invariance

Let $\chi(x)$ be an arbitrary scalar fct. of space-time.

def. 1: A transformation of the potential $A^\mu(x)$ according to

$$\boxed{A^\mu(x) \to A'^\mu(x) = A^\mu(x) - \partial^\mu \chi(x)}$$

is called a gauge transformation.

proposition: The action from §1.1 eqn 3 is invariant under gauge trafos.

proof: $F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$

$$\to \partial^\mu A^\nu(x) - \cancel{\partial^\mu \partial^\nu \chi(x)} - \partial^\nu A^\mu(x) + \cancel{\partial^\nu \partial^\mu \chi(x)}$$

$= F^{\mu\nu}(x)$ $\to$ the field tensor is invariant

$\to$ S_{vac} is invariant

$$S_{int} = -\frac{1}{c}\int d^4x\, j_\mu(x) A^\mu(x) \to S_{int} + \frac{1}{c}\int d^4x\, j_\mu(x)\partial^\mu \chi(x)$$

$$\overset{\text{part. int}}{=} S_{int} - \frac{1}{c}\int d^4x \underbrace{\left(\partial^\mu j_\mu(x)\right)}_{=0 \text{ by } §2.1} \chi(x) = S_{int}$$

$\to$ S_{int} is invariant $\square$

remark: (1) The potential is not unique. This is a result of the fact that $F^{\mu\nu}$ depends only on derivatives of A^μ.

(2) We may choose a gauge trafo to enforce a particular condition on A^μ ("fixing the gauge").

Problem 8: Coulomb gauge

corollary: A^μ can be chosen such that $\boxed{\partial_\mu A^\mu(x) = 0}$ "Lorenz gauge" (no tz!)

proof: choose χ such that it solves the PDE $\partial^\mu \partial_\mu \chi(x) = \partial_\mu A^\mu(x)$

$$\to \partial_\mu A'^\mu(x) = \partial_\mu A^\mu(x) - \partial_\mu \partial^\mu \chi(x) = 0 \quad \square$$

remark: (3) $\partial_\mu A^\mu$ is a Lorentz scalar $\to$ the Lorenz gauge is Lorentz invariant

§3 Electric and magnetic fields

3.1 The field tensor in terms of Euclidean vector fields

PHYS 610 ch I §5.4 -> As an antisymmetric Minkowski tensor, $F_{\mu\nu}$ has the form

$$\boxed{F_{\mu\nu} = \left(\begin{array}{c|ccc} 0 & E_x & E_y & E_z \\ \hline -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{array}\right)} = \left(\begin{array}{c|c} 0 & \vec{E} \\ \hline -\vec{E} & B_{ij} \end{array}\right)$$

LL's notation: $\vec{E} = (E^1, E^2, E^3) \equiv (E_x, E_y, E_z)$! $\neq (E_1, E_2, E_3)$ same for $\vec{B}$

with $\vec{E}(x) = (E_x(x), E_y(x), E_z(x))$ a Euclidean vector field

and $\vec{B}(x) = (B_x(x), B_y(x), B_z(x))$ a Euclidean pseudovector field

def. 1: (a) $\vec{E}(x)$ is called electric field, and $\vec{B}(x)$ is called magnetic field.

(b) The antisymmetric Euclidean tensor

$$B_{ij} = \begin{pmatrix} 0 & -B_z & B_y \\ B_z & 0 & -B_x \\ -B_y & B_x & 0 \end{pmatrix} = B^{ij} = -\varepsilon_{ij\lambda} B^\lambda$$

is called magnetic field tensor.

Week 2 (short due to MLK Day) w3 ... §5,6,7,8)

remark: (1) $F^{\mu\nu} = g^{\mu\kappa} g^{\nu\lambda} F_{\kappa\lambda} = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_\mu \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_\nu F_{\mu\nu} = \left(\begin{array}{c|c} 0 & -\vec{E} \\ \hline \vec{E} & B^{ij} = B_{ij} \end{array}\right)$

(1') $\varepsilon^{ij\lambda} = g^{i\kappa} g^{j\lambda} g^{\lambda\gamma} \varepsilon_{\kappa\lambda\gamma} = -\varepsilon_{ij\lambda} = -\varepsilon_i{}^{j\lambda} = \varepsilon_{ij}{}^{\lambda}$ etc.

(1'') $\varepsilon^{ij\lambda} \varepsilon_{i\ell m} = -(\delta^j_\ell \delta^\lambda_m - \delta^j_m \delta^\lambda_\ell)$ (NB the overall sign!)

PHYS 610 ch I §5.4 -> A^μ has the form

$$\boxed{A^\mu(x) = (\varphi(x), \vec{A}(x))}$$

$(\varphi(x) \equiv A_0(x) = A^0(x))$

and $\vec{A}(x)$ a Euclidean vector field

def. 2 $\varphi(x)$ is called scalar potential, and $\vec{A}(x)$ is called vector potential

remark: (2) $j^\mu(x) = (c\rho(x), \vec{j}(x))$ with ρ the charge density and $\vec{j}$ the current density: see §2.1.

3.2 Maxwell's equations

Write the field eqs from §1.3 in terms of $\vec{E}$ and $\vec{B}$.

§1.3 (**): $\quad \partial_\mu \varepsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda} = 0$

$\underline{\nu = 0}$: $\quad \underline{0} = 2\left(-\partial_1 F_{23} + \partial_2 F_{13} - \partial_3 F_{12}\right) = 2\left(\partial_1 B_x + \partial_2 B_y + \partial_3 B_z\right) = \underline{\vec{\nabla}\cdot\vec{B}}$

$\underline{\nu = 1}$: $\quad \underline{0} = 2\left(\partial_0 F_{23} - \partial_2 F_{03} + \partial_3 F_{02}\right) = -\frac{1}{c}\partial_t B_x - \partial_2 E_z + \partial_3 E_y$

$$= \underline{-\frac{1}{c}\partial_t B_x - (\vec{\nabla}\times\vec{E})_x}$$

and cyclic

proposition 1: §1.3 (**) is equivalent to

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{B}(x) &= 0 \\ \frac{1}{c}\partial_t\vec{B}(x) + \vec{\nabla}\times\vec{E}(x) &= 0 \end{aligned}} \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

remark: (1) These are 4 homogeneous PDEs for the six field components $E_{x,y,z}$, $B_{x,y,z}$.

Now consider the EL eq., §1.3 (*):

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$$

$\underline{\nu = 0}$: $\quad \underbrace{\partial_0 F^{00}}_{=0} + \partial_i F^{i0} = \frac{4\pi}{c} j^0 \rightsquigarrow \frac{\partial}{\partial x^i} E^i = \frac{4\pi}{c} c\rho \rightsquigarrow \underline{\vec{\nabla}\cdot\vec{E} = 4\pi\rho}$

$\underline{\nu = 1}$: $\quad \partial_0 F^{01} + \partial_i F^{i1} = \frac{4\pi}{c} j^1 \rightsquigarrow -\frac{1}{c}\partial_t E_x + \partial_2 B_z - \partial_3 B_y = \frac{4\pi}{c} j^x$

and cyclic $\qquad \rightsquigarrow \underline{-\frac{1}{c}\partial_t E_x + (\vec{\nabla}\times\vec{B})_x = \frac{4\pi}{c} j^x}$

proposition 2: §1.3 (*) is equivalent to

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E}(x) &= 4\pi\rho(x) \\ -\frac{1}{c}\partial_t\vec{E}(x) + \vec{\nabla}\times\vec{B}(x) &= \frac{4\pi}{c}\vec{j}(x) \end{aligned}} \qquad \begin{aligned} &(3) \\ &(4) \end{aligned}$$

(5) Units: We use cgs units: $[\text{charge}] = \text{esu} = g^{1/2} cm^{3/2}/s$

$$[\rho] = g^{1/2}/cm^{3/2}s \,,\quad [\vec{j}] = g^{1/2}/cm^{1/2}s^2$$

$$[\vec{E}] = g^{1/2}/cm^{1/2}s \,,\quad [\vec{B}] = g^{1/2}/cm^{1/2}s$$

In SI units, $[\text{charge}] = C$, $[\rho] = C/m^3$, $[\vec{j}] = C/m^2 s$

$[\vec{E}] = N/C$, $[\vec{B}] = N/Am$ (Note: $C = As$)

and the M-eqs (2)-(4) read →

$$\boxed{\partial_t \vec{B} + \vec{\nabla}\times\vec{E} = 0} \quad (2')$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho} \quad (3')$$

$$\boxed{-\partial_t \vec{E} + \frac{1}{\mu_0\varepsilon_0}\vec{\nabla}\times\vec{B} = \frac{1}{\varepsilon_0}\vec{j}} \quad (4')$$

where $\varepsilon_0 = 8.85\times10^{-12}\, C^2/Nm^2$ "permittivity of vacuum"

$\mu_0 = 4\pi\times10^{-7}\, N/A^2$ "permeability of vacuum"

and $\mu_0\varepsilon_0 = 1/c^2 = 1.11\times10^{-17}\, s^2/m^2$

remark: (2) Eqs (1)–(4) are called Maxwell's eqs. Their solutions determine physical field configurations for given charge and current densities.

(3) Eqs (1)–(4) are equivalent to Eqs. (*), (**) in §1.3.

(4) $\vec{E}$ and $\vec{B}$ are Euclidean vector fields. ~> The Lorentz invariance of (1)–(4) is far from obvious, whereas the Lorentz invariance of (*), (**) is.

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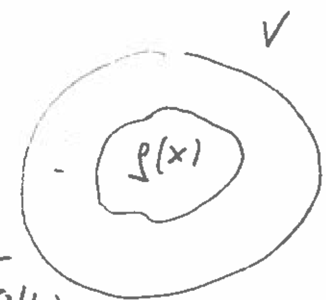
2.2 Discussion of Maxwell's equations



Consider a localized charge density ρ within a volume V with surface ∂V.

M-eq. (2) ~> $\int_V d^3x\, \vec{\nabla}\cdot\vec{E}(\vec{x},t) = 4\pi \int_V d^3x\, \rho(\vec{x},t) = 4\pi Q(t)$ = total charge

$\int_{\partial V} d\vec{S}\cdot\vec{E}(\vec{x},t)$ = flux of $\vec{E}$ through the surface ∂V.

(0) This is called Gauss's law

remark: (1) Electric charges are the sources of electric fields.

M-eq. (1) ~> $0 = \int_V d^3x\, \vec{\nabla}\cdot\vec{B}(\vec{x},t) = \int_{\partial V} d\vec{S}\cdot\vec{B}(\vec{x},t)$ = flux of $\vec{B}$ through any closed surface!

remark: (2) The magnetic field has no sources! Equivalent statement: There is no magnetic charge (aka magnetic monopole).

(3) In the manifestly Lorentz-invariant formulation of §1 this asymmetry manifests itself in the asymmetry of the field eqs:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu, \quad \text{but} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

Consider a bounded surface S with boundary (S).

M-eq. (2) $\rightarrow$

$$-\frac{1}{c}\int_S d\vec{S}\cdot\partial_t\vec{B}(\vec{x},t) = \int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{E})(\vec{x},t) \overset{\text{Stokes}}{=} \oint_{(S)} d\vec{l}\cdot\vec{E}(\vec{x},t) = \text{circulation of } \vec{E} \text{ around } (S)$$

$$\parallel$$

$$-\frac{1}{c}\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}(\vec{x},t) = -\frac{1}{c}\frac{d}{dt}\Phi(t) = \text{time derivative of flux of } \vec{B} \text{ through the surface } S$$

remark: (4) This is called Faraday's law of induction

(5) Consider a closed $\vec{E}$-field line

Then $\oint d\vec{l}\cdot\vec{E} > 0 \rightarrow \dot{\Phi}(t) < 0$

$\rightarrow$ In a static $\vec{B}$-field (in particular for $\vec{B}=0$) there can be no closed $\vec{E}$-field lines!

M-eq. (4) $\rightarrow$

$$\int_S d\vec{S}\cdot(\vec{\nabla}\times\vec{B}(\vec{x},t)) = \frac{4\pi}{c}\underbrace{\int_S d\vec{S}\cdot\vec{j}(\vec{x},t)}_{J} + \frac{1}{c}\int_S d\vec{S}\cdot\partial_t\vec{E}(\vec{x},t)$$

$$\text{circulation of } \vec{B} = \oint_{(S)} d\vec{l}\cdot\vec{B}(\vec{x},t) = \frac{4\pi}{c}J + \frac{1}{c}\frac{d}{dt}\int_S d\vec{S}\cdot\vec{E}(\vec{x},t)$$

$$= \frac{4\pi}{c}\times \text{total current} + \frac{1}{c}\times\text{"displacement current"}$$

remark: (6) This is called Ampère-Maxwell law

(7) Currents induce $\vec{B}$-fields, and vice versa

(8) For static fields, we have Ampère's law

$$\oint d\vec{l}\cdot\vec{B} = \frac{4\pi}{c}J$$

The displacement current was later added by Maxwell

3.4 Relation between fields and potentials

§3.1 → $E^i = -F^{0i} = -\partial^0 A^i + \partial^i A^0$

$$= -\partial_0 A^i - \partial_i A^0 = -\frac{1}{c}\partial_t A^i - \partial_i \varphi$$

$$\rightarrow \quad \boxed{\vec{E}(\vec{x},t) = -\vec{\nabla}\varphi(\vec{x},t) - \frac{1}{c}\partial_t \vec{A}(\vec{x},t)}$$

remark: (1) In general, both $\vec{A}$ and φ determine $\vec{E}$.

(2) Gauge transfo: $A^\mu \rightarrow A^\mu - \partial^\mu \chi \quad \rightarrow \quad \varphi \rightarrow \varphi - \frac{1}{c}\partial_t \chi$, $\quad \vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi$

$$\rightarrow \quad \vec{E} \rightarrow \vec{E} + \vec{\nabla}\frac{1}{c}\partial_t \chi - \frac{1}{c}\partial_t \vec{\nabla}\chi = \vec{E}$$

$\vec{E}$ is unaffected

§3.1 → $F_{12} = (\partial_1 A_2 - \partial_2 A_1) = (\vec{\nabla}\times\vec{A})_3 = -(\vec{\nabla}\times\vec{A})^3$

$\quad\quad = -B_3 = -B^3$ and cyclic

$$\rightarrow \quad \boxed{\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}(\vec{x},t)}$$

remark: (3) $\vec{B}$ invariant under gauge transfos, since $\vec{\nabla}\times(\vec{\nabla}\chi) = 0$

3.5 Charges in electromagnetic fields

Attitude so far: Field eqs determine fields for given charges and currents.

Question: For given fields, what is their influence on a point charge?

Let a point particle with charge e be at point $\vec{\xi}(t)$ with velocity $\dot{\vec{\xi}}(t) =: \vec{v}(t)$.

$$\rightsquigarrow \quad \rho(\vec{x},t) = e\,\delta(\vec{x}-\vec{\xi}(t)) \qquad \text{charge density}$$

$$\vec{j}(\vec{x},t) = \rho(\vec{x},t)\,\vec{v}(t) \qquad \text{current density}$$

$$j^\mu = (c\rho,\vec{j}),\; j_\mu = (c\rho,-\vec{j}) \qquad \text{4-current} \qquad A^\mu = (\varphi,\vec{A}) \quad \text{4-potential}$$

§1.1 axiom (3) $\rightsquigarrow$ $\underline{S_{int}} = -\frac{1}{c}\int d^4x\, j_\mu(x)\, A^\mu(x)$

$$= -\frac{1}{c}c\int dt \int d\vec{x}\, \cancel{c}\rho(\vec{x},t)\,\varphi(\vec{x},t) + \frac{1}{c}c\int dt\int d\vec{x}\, \vec{j}(\vec{x},t)\cdot\vec{A}(\vec{x},t)$$

$$= -c\int dt\; e\,\varphi(\vec{\xi},t) + e\int dt\; \vec{v}\cdot\vec{A}(\vec{\xi},t)$$

Now consider the Lagrangian of the point particle, $L_{int}(\vec{\xi},\dot{\vec{\xi}},t)$, which is related to S_{int} via $S_{int} = c\int dt\, L_{int}(\vec{\xi},\dot{\vec{\xi}},t)$

$$\rightsquigarrow \quad \boxed{L_{int}(\vec{\xi},\dot{\vec{\xi}},t) = -e\,\varphi(\vec{\xi},t) + \frac{e}{c}\,\vec{v}\cdot\vec{A}(\vec{\xi},t)} \quad \text{where} \quad \vec{v} = \dot{\vec{\xi}}$$

Remark: (1) φ and $\vec{A}$ are examples of the scalar and vector potential, respectively, familiar from Mechanics, see §3.5a.

Problem 9

particle in $\vec{E}$ and $\vec{B}$ fields

(2) §1.1 axiom (3) is consistent with the axioms of Mechanics.

Problem 10

harm. osc. in $\vec{B}$-field

(3) L_{int} must be augmented by the Lagrangian L_0 for a free particle.

Field eqs are Lorentz invariant $\rightsquigarrow$ L_0 must be the Einsteinian L_0 for consistency (although Galilean L_0 is also approx. for $|\vec{v}| \ll c$).

Problem 11

[illegible] fields

3.5c Digression: Potentials in Mechanics, and Newton's 2nd Law

Let $L_0(\vec{v})$ be the Lagrangian for a free point particle. Consider the interaction of the particle with its environment.

axiom: The influence of the environment on the particle is described by (a) a scalar potential $U(\vec{x},t)$ and (b) a vector potential $\vec{V}(\vec{x},t)$ such that the Lagrangian is

$$\boxed{L(\vec{x},\vec{v},t) = L_0(\vec{v}) - U(\vec{x},t) + \vec{v}\cdot\vec{V}(\vec{x},t)}$$

remark: (1) L_0 may describe a Galilean particle, $L_0^G = \frac{m}{2}\vec{v}^2$, or an Einsteinian one, $L_0^E = -mc^2\sqrt{1-\vec{v}^2/c^2}$.

(2) U and $\vec{V}$ are determined either by experiment, or by another theory, not by mechanics

example: (1) Particle in a gravitational field. U determined by experiment (Kepler) or a more general theory (GR, Einstein). $\vec{V}=0$.

(2) Charged particle in an electromagnetic field. U and $\vec{V}$ determined by a different theory (Maxwell), see §3.5: $U(\vec{x},t) = -e\varphi(\vec{x},t)$, $\vec{V}(\vec{x},t) = \frac{e}{c}\vec{A}(\vec{x},t)$.

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def.: In the presence of a vector potential we define the particle momentum by $\boxed{\vec{p} = \partial L_0/\partial\vec{v}}$

remark: (3) $\vec{\pi} := \partial L/\partial\vec{v}$ is sometimes called generalized mom lma: $\boxed{F_{\alpha\beta}F^{\alpha\beta} = -2(\vec{E}^2 - \vec{B}^2)}$

proof: $F_{\alpha\beta}F^{\alpha\beta} = F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}$

i^{th} column

$F_{0i} = E^i = -E_i$

0^{th} row

$F^{0i} = -E^i$

$$= 0 + E_iE^i + E_iE^i + B_{ij}B^{ij}$$

$$= -2\vec{E}^2 + \varepsilon_{ijl}B^l\varepsilon^{ijl}B_e$$

§3.1 remark (1')

$$= -2\vec{E}^2 - (\delta_j{}^j\delta_l{}^e - \delta_j{}^e\delta_l{}^j)B^lB_e$$

$$= -2\vec{E}^2 - (3\delta_l{}^e - \delta_l{}^e)B^lB_e$$

$$= -2\vec{E}^2 - 2B^lB_l$$

$$= -2\vec{E}^2 + 2\vec{B}^2$$

proposition: The equation of motion takes the form

$$\boxed{\frac{d}{dt}\vec{p}(\vec{x},t) = \vec{F}^{(1)}(\vec{x},t) + \vec{F}^{(2)}(\vec{x},\vec{v},t)}$$ Newton's 2nd law

with $\boxed{\vec{F}^{(1)}(\vec{x},t) = -\vec{\nabla}U(\vec{x},t) - \partial_t\vec{V}(\vec{x},t)}$ a velocity-independent force

and $\boxed{\vec{F}^{(2)}(\vec{x},t) = \vec{v}\times\left(\vec{\nabla}\times\vec{V}(\vec{x},t)\right)}$ a velocity-dependent force

proof: Euler-Lagrange ->

$$\frac{d}{dt}\frac{\partial L}{\partial v_i} = \frac{d}{dt}\pi_i = \frac{d}{dt}p_i + \frac{d}{dt}V_i \overset{EL}{=} \frac{\partial L}{\partial x_i} = -\partial_i U + v_j\partial_i V_j$$

$$\rightarrow \frac{d}{dt}p_i = \underbrace{-\partial_i U - \partial_t V_i}_{=F^{(1)}_i} - (\partial_j V_i)v_j + v_j\partial_i V_j$$

$$\begin{aligned}(\vec{v}\times(\vec{\nabla}\times\vec{V}))_i &= \varepsilon_{ijl}v_j\varepsilon_{lmn}\partial_m V_n \\ &= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})v_j\partial_m V_n \\ &= v_j\partial_i V_j - v_j\partial_j V_i\end{aligned}$$

Week 3
Problems 3
(8,9,10,11)

$$= -\partial_i U - \partial_t V_i + (\vec{v}\times(\vec{\nabla}\times\vec{V}))_i = F^{(1)}_i + F^{(2)}_i \qquad \square$$

3.6 Poynting's theorem

Consider the continuity eq. from §2.3 for $\mu=0$:

$$T^{00} = \frac{-1}{4\pi}F^{0\alpha}F^0{}_\alpha + \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} = \frac{-1}{4\pi}F^{0i}F^0{}_i - \frac{1}{8\pi}(\vec{E}^2-\vec{B}^2) = \frac{-1}{4\pi}F^{0i}F^{0k}\delta_{ki} - \frac{1}{8\pi}(\vec{E}^2-\vec{B}^2)$$

$$= \frac{-1}{4\pi}(-E^i)(-E_i) - \frac{1}{8\pi}(\vec{E}^2-\vec{B}^2) = \frac{1}{4\pi}\vec{E}^2 - \frac{1}{8\pi}(\vec{E}^2-\vec{B}^2) = \frac{1}{8\pi}(\vec{E}^2+\vec{B}^2)$$

$$T^{01} = \frac{-1}{4\pi}F^{0\alpha}F^1{}_\alpha = \frac{-1}{4\pi}F^{0j}F^1{}_j = \frac{-1}{4\pi}(-E_y B_z + E_z B_y) = \frac{1}{4\pi}(\vec{E}\times\vec{B})^1$$ and cyclic

$$\rightarrow \partial_\nu T^{0\nu} = \frac{1}{c}\partial_t\frac{1}{8\pi}(\vec{E}^2+\vec{B}^2) + \vec{\nabla}\cdot\frac{1}{4\pi}(\vec{E}\times\vec{B})$$ this is the lhs of the continuity eq

$$F^{0\nu}j_\nu = F^{0i}j_i = \vec{E}\cdot\vec{j}$$ this is the rhs

Define $\underline{u(\vec{x},t) := \frac{1}{8\pi}\left(\vec{E}^2(\vec{x},t) + \vec{B}^2(\vec{x},t)\right)}$ <u>energy density</u> of the field (see below).

$\underline{\vec{P}(\vec{x},t)} := \underline{\frac{c}{4\pi}\, \vec{E}(\vec{x},t) \times \vec{B}(\vec{x},t)}$ "<u>Poynting vector</u>" = energy current density (see below)

-> In terms of $\vec{E}$ and $\vec{B}$ the continuity eq. takes the form

$$\partial_t u + \vec{\nabla}\cdot\vec{P} = -\vec{E}\cdot\vec{j}$$

$$\boxed{\partial_t u(\vec{x},t) + \vec{\nabla}\cdot\vec{P}(\vec{x},t) = -\vec{E}(\vec{x},t)\cdot\vec{j}(\vec{x},t)}$$ "<u>Poynting's theorem</u>"

<u>remark</u>: (1) For $\vec{j}=0$, this expresses local energy conservation. It is analogous to §2.1 with $\rho \to u$ (energy density) and $\vec{j} \to \vec{P}$ (energy current density).

(2) §2.5 -> $\vec{v}\cdot\vec{F} = e\vec{v}\cdot\vec{E}$ = work per time done by the fields on a charge e.

-> $\vec{j}\cdot\vec{E} = \frac{1}{V} e\vec{v}\cdot\vec{E}$ = work done per time and volume = power density

-> For $\vec{j} \neq 0$, P's theorem still expresses energy conservation:

$$\boxed{\text{energy change} = -\text{energy transported by energy current} - \text{work done by fields on charges}}$$

DO! (3) We still need to show that $u(\vec{x},t)$ can be sensibly interpreted as the energy density of the field. Let $\vec{j}$ be the current density due to just one particle, as in §3.5 (for many particles, just sum over them)

$\to \int d\vec{x}\, \vec{j}\cdot\vec{E} = \int d\vec{x}\, \vec{E}(\vec{x},t)\cdot e\vec{v}\cdot\delta(\vec{x}-\vec{y}) = e\vec{v}\cdot\vec{E}(\vec{y})$

where $\vec{y}$ is the position of the particle.

Consider a nonrelativistic particle for simplicity (the argument also holds in the relativistic case, see Problem 12)

Problem 12 energy density

$\to E_{ki} = \frac{m}{2} v^2$ kinetic energy

$\to \frac{d}{dt} E_{ki} = m\vec{v}\cdot\frac{d\vec{v}}{dt} = \vec{v}\cdot\frac{d}{dt}\vec{p} \overset{\S 3.5 \text{ remark (4)}}{=} e\vec{v}\cdot\vec{E}$

Now integrate Poynting's theorem over all of space:

$$\frac{d}{dt}\int d\vec{x}\, u(\vec{x},t) + \underbrace{\int d\vec{x}\, \vec{\nabla}\cdot\vec{P}(\vec{x},t)}_{=\int d\vec{s}\,\vec{P} = 0 \text{ since } \vec{P}=0 \text{ at } \infty} = -\int d\vec{x}\, \vec{j}\cdot\vec{E} = -e\vec{v}\cdot\vec{E} = -\frac{d}{dt}E_{ki}$$

DO

$\to \frac{d}{dt}(U + E_{ki}) = 0$ where $U = \int d\vec{x}\, u(\vec{x},t)$

$\to U$ must be the field energy, since the energy of the particle plus the energy of the field must be conserved

$\to u(\vec{x},t)$ must be the energy density of the field

(4) Integrate over a finite volume $\to$ Energy may change due to an energy current across the volume boundary

$\to \vec{P}$ should be interpreted as the energy current density of the field.

(5) The remaining components of the continuity eq. from §3.3,

$\partial_\nu T^{i\nu} = -\frac{1}{c} F^{i\nu} j_\nu$

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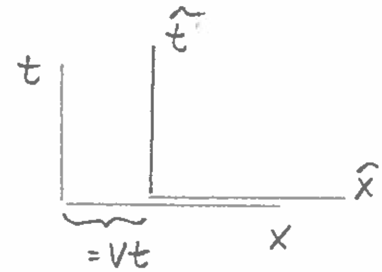
express the fact that the energy current density is also conserved.

§4 Lorentz transformation of the fields

4.1 Physical interpretation of a Lorentz boost



Consider two inertial frames, CS and $\widehat{CS}$

Let $\widehat{CS}$ move with respect to CS with a constant velocity $\vec{V} = (V,0,0)$.

Problem 610 §21,23 → The trafo from CS to $\widehat{CS}$ is accomplished by a Lorentz boost

$$c\hat{t} = ct\cosh\phi - x\sinh\phi$$
$$\hat{x} = ct\sinh\phi - x\cosh\phi$$

Consider the origin in $\widehat{CS}$ as viewed in CS. Then $\hat{x} = 0$

$$\to \quad x\cosh\phi = -ct\sinh\phi$$

$$\to \quad V = x/t = -c\tanh\phi$$

$$\to \quad \underline{\sinh\phi} = \frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}} = \underline{\frac{V/c}{\sqrt{1-V^2/c^2}}}, \quad \underline{\cosh\phi} = \sqrt{1+\sinh^2\phi} = \underline{\frac{1}{\sqrt{1-V^2/c^2}}}$$

Remark: (1) For $c \to \infty$ we recover the Galileo trafo

$$\hat{x} = x + Vt, \quad \hat{t} = t$$

→ A Lorentz boost along the x-axis is given by

Problem 13
Addition of velocities
cf. §5

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\phi & \sinh\phi & & \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

with $\cosh\phi = \gamma$, $\sinh\phi = \frac{V}{c}\gamma$, where $\gamma := \frac{1}{\sqrt{1-V^2/c^2}}$

4.2 Transformation of $\vec E$ and $\vec B$ under a Lorentz boost

Consider the field tensor $F^{\mu\nu}$ in CS. The field tensor $\tilde F^{\mu\nu}$ in $\tilde{CS}$ is

$$\boxed{\tilde F^{\mu\nu} = D^\mu{}_\alpha D^\nu{}_\lambda F^{\alpha\lambda}} \qquad \text{and} \qquad \tilde x^\mu = D^\mu{}_\nu x^\nu$$

Now let $D^\mu{}_\nu$ be a Lorentz boost.

$$\rightsquigarrow \tilde F^{\mu\nu} = \left(D F D^T\right)^{\mu\nu} \overset{§3.1}{=} \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi & & \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} E_x \sinh\phi & -E_x\cosh\phi & -E_y\cosh\phi - B_z\sinh\phi & -E_z\cosh\phi + B_y\sinh\phi \\ E_x\cosh\phi & -E_x\sinh\phi & -E_y\sinh\phi - B_z\cosh\phi & -E_z\sinh\phi + B_y\cosh\phi \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi & & 0 \\ \sinh\phi & \cosh\phi & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ E_x & 0 & \cdot & \cdot \\ E_y\cosh\phi + B_z\sinh\phi & E_y\sinh\phi + B_z\cosh\phi & 0 & \cdot \\ E_z\cosh\phi - B_y\sinh\phi & E_z\sinh\phi - B_y\cosh\phi & B_x & \cdot \end{pmatrix}$$

problem 14

Lor-invariance der Gleichungen sofort WG §4.1

$$\equiv \begin{pmatrix} 0 & -\tilde E_x & -\tilde E_y & -\tilde E_z \\ \tilde E_x & 0 & -\tilde B_z & \tilde B_y \\ \tilde E_y & \tilde B_z & 0 & -\tilde B_x \\ \tilde E_z & -\tilde B_y & \tilde B_x & 0 \end{pmatrix} \rightsquigarrow \boxed{\begin{aligned} \tilde E_x &= E_x \\ \tilde E_y &= E_y\cosh\phi + B_z\sinh\phi \\ \tilde E_z &= E_z\cosh\phi - B_y\sinh\phi \\ \tilde B_x &= B_x \\ \tilde B_y &= B_y\cosh\phi - E_z\sinh\phi \\ \tilde B_z &= B_z\cosh\phi + E_y\sinh\phi \end{aligned}} \quad (*)$$

remark: (1) The field eqs were formulated in terms of Minkowski tensors $\rightsquigarrow$ Their Lorentz invariance is guaranteed. (*) reflects the Lorentz invariance of M's eqs, which ...

remark: (2) let $V/c \ll 1$ and keep terms to $O(V/c)$.

$\rightarrow$ $\cosh\phi \simeq 1$, $\sinh\phi \simeq V/c$

$\rightarrow$ $\underline{\hat{\vec{E}} \simeq \vec{E} - \frac{1}{c}\vec{V}\times\vec{B} + O(V^2/c^2)}$, $\underline{\hat{\vec{B}} = \vec{B} + \frac{1}{c}\vec{V}\times\vec{E} + O(V^2/c^2)}$

(3) let $\vec{E} = 0$, i.e., no $\vec{E}$-field in CS.

$\rightarrow$ $\hat{\vec{E}} = -\frac{1}{c}\vec{V}\times\vec{B}$ $\rightarrow$ in $\hat{\text{CS}}$ there is an $\vec{E}$-field, as long as $\vec{B} \neq 0$!

4.3 Lorentz invariants

From the field tensor $F^{\mu\nu}$ we can form two Lorentz scalars:

$$\boxed{\mathfrak{F}^{(s)} := -\frac{1}{2}F^{\mu\nu}F_{\mu\nu} \quad , \quad \mathfrak{F}^{(p)} := -\frac{1}{8}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}}$$

remark: (1) $\mathfrak{F}^{(s)}$ is a scalar $\rightarrow$ $\hat{\mathfrak{F}}^{(s)} = \mathfrak{F}^{(s)}$ = const in all inertial frames

$\mathfrak{F}^{(p)}$ is a pseudoscalar $\rightarrow$ $|\mathfrak{F}^{(p)}|$ = const in all inertial frames

but $\hat{\mathfrak{F}}^{(p)} = \det\Lambda \cdot \mathfrak{F}^{(p)}$

§3.6 tensor $\rightarrow$ $\underline{\underline{\mathfrak{F}^{(s)}}} = -\frac{1}{2}(-2)(\vec{E}^2 - \vec{B}^2)$

$= \underline{\underline{\vec{E}^2 - \vec{B}^2}}$

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$$\underline{\mathfrak{F}^{(p)}} = \frac{1}{8}\Big[\varepsilon^{0123}F_{01}F_{23} + \varepsilon^{0132}F_{01}F_{32} + \varepsilon^{0213} + \varepsilon^{0231} + \varepsilon^{0312} + \varepsilon^{0321} + (3\times 6 = 18 \text{ other terms})\Big]$$

$$= \frac{1}{4}\Big[\varepsilon^{0123}F_{01}F_{23} + \varepsilon^{0213}F_{02}F_{13} + \varepsilon^{0312}F_{03}F_{12} + 9 \text{ other terms}\Big]$$

$$= \frac{1}{4}\left[-E_x B_x - E_y B_y - E_z B_z\right] \times 4 = -\vec{E}\cdot\vec{B}$$

proposition. The field combinations

$$\boxed{F^{(1)} = \vec{E}^2 - \vec{B}^2, \quad F^{(2)} = \vec{E}\cdot\vec{B}}$$

transform as a scalar ($F^{(1)}$) and pseudoscalar ($F^{(2)}$), respectively, under Lorentz transformations, i.e., their absolute values have the same value in all inertial frames.

remark: (2) If $\vec{E} \perp \vec{B}$ in some inertial frame, then $\vec{E} \perp \vec{B}$ in all other inertial frames.

Problem 15

L §25 Problem: Lorentz trafo of fields

(3) Ditto if $\vec{E}^2 = \vec{B}^2$ in some frame

§5 The superposition principle of Maxwell theory

5.1 Field solutions

proposition 1: Let $\rho^{(\lambda)}(x), \vec{j}^{(\lambda)}(x)$ $(\lambda = 1,2)$ be two charge and current densities, let $\vec{E}^{(\lambda)}(x), \vec{B}^{(\lambda)}(x)$ be solutions of M's eqs for $\rho^{(\lambda)}, \vec{j}^{(\lambda)}$, and let $\lambda^{(\lambda)} \in \mathbb{R}$. Then

$$\vec{E} = \lambda^{(1)}\vec{E}^{(1)} + \lambda^{(2)}\vec{E}^{(2)}, \quad \vec{B} = \lambda^{(1)}\vec{B}^{(1)} + \lambda^{(2)}\vec{B}^{(2)}$$

are solutions for $\lambda^{(1)}\rho^{(1)} + \lambda^{(2)}\rho^{(2)}$, $\lambda^{(1)}\vec{j}^{(1)} + \lambda^{(2)}\vec{j}^{(2)}$

proof: $\vec{\nabla}\cdot\vec{E} - 4\pi\rho = \underbrace{\vec{\nabla}\cdot\vec{E}^{(1)} - 4\pi\rho^{(1)}}_{=0} + \underbrace{\vec{\nabla}\cdot\vec{E}^{(2)} - 4\pi\rho^{(2)}}_{=0} = 0$

etc.

remark: (1) This is obviously true since the theory is linear!

(2) If the action contained terms higher than quadratic in $F^{\mu\nu}$ it would not be true.

(3) A field theory that leads to linear field eqs is called Gaussian or free

corollary 1: let $\vec{E}^{(\lambda)}(x)$, $\vec{B}^{(\lambda)}(x)$ be solutions for $\rho^{(\lambda)}(x), \vec{j}^{(\lambda)}(x)$ with $\lambda \in \mathbb{R}$, and let $\lambda(\lambda) \in \mathbb{R}$ be a sufficiently well behaved fct. of λ. Then

$$\vec{E}(x) = \int d\lambda \, \lambda(\lambda) \, \vec{E}^{(\lambda)}(x) \, , \quad \vec{B}(x) = \int d\lambda \, \lambda(\lambda) \, \vec{E}^{(\lambda)}(x)$$

are solutions for $\rho(x) = \int d\lambda \, \lambda(\lambda) \rho^{(\lambda)}(x)$, $\vec{j}(x) = \int d\lambda \, \lambda(\lambda) \vec{j}^{(\lambda)}(x)$

proof: Generalize prop 1 to $\alpha = 1, \dots, N$ and let $N \to \infty$.

remark: (4) This can obviously be generalized to $\vec{E}^{(\vec{\lambda})}(x)$ with $\vec{\lambda} \in \mathbb{R}^3$.

corollary 2: The most general solution of M's eqs. is obtained as

$$\vec{E}(x) = \vec{E}^{(0)}(x) + \vec{E}^{(p)}(x) \, , \quad \vec{B}(x) = \vec{B}^{(0)}(x) + \vec{B}^{(p)}(x)$$

where $\vec{E}^{(0)}, \vec{B}^{(0)}$ are the most general solution in vacuum (i.e., for $\rho = \vec{j} = 0$) and $\vec{E}^{(p)}, \vec{B}^{(p)}$ is a particular solution of the eqs in the presence of $\rho, \vec{j}$.

proof: let $\vec{E}, \vec{B}$ be any solution for $\rho, \vec{j}$, and let $\vec{E}^{(p)}, \vec{B}^{(p)}$ be a particular solution. Prop. 1 $\leadsto$

$$\vec{E}^{(0)} := \vec{E} - \vec{E}^{(p)}, \quad \vec{B}^{(0)} = \vec{B} - \vec{B}^{(p)}$$

is a solution for $\rho = 0$, $\vec{j} = 0$.

Conversely, if $\vec{E}^{(0)}$ is a solution for $\rho = \vec{j} = 0$, and $\vec{E}^{(p)}, \vec{B}^{(p)}$ is some solution for $\rho, \vec{j}$, then $\vec{E} = \vec{E}^{(0)} + \vec{E}^{(p)}$, $\vec{B} = \vec{B}^{(0)} + \vec{B}^{(p)}$ is a solution for $\rho, \vec{j}$. $\square$

5.2 Complex solutions

All physical solutions of M's eqs consist of real fields $\vec{E}$ and $\vec{B}$. However, it sometimes is convenient to find complex solutions and take the real part.

proposition I: Let $\vec{E}, \vec{B}$ be complex solutions for complex $\rho, \vec{j}$. Then $\vec{E}^*, \vec{B}^*$ are solutions for $\rho^*, \vec{j}^*$.

proof: $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot \mathrm{Re}\,\vec{E} + i\vec{\nabla}\,\mathrm{Im}\,\vec{E} = 4\pi\,\mathrm{Re}\,\rho + i4\pi\,\mathrm{Im}\,\rho$

$$\rightsquigarrow \left.\begin{array}{l} \vec{\nabla}\cdot \mathrm{Re}\,\vec{E} = 4\pi\,\mathrm{Re}\,\rho \\ \vec{\nabla}\cdot \mathrm{Im}\,\vec{E} = 4\pi\,\mathrm{Im}\,\rho \end{array}\right\} \rightsquigarrow \vec{\nabla}(\mathrm{Re}\,\vec{E} - i\,\mathrm{Im}\,\vec{E}) = 4\pi(\mathrm{Re}\,\rho - i\,\mathrm{Im}\,\rho);$$

etc.

remark: (1) This equation is true because of linearity.

corollary I: Let $\vec{E}, \vec{B}$ be complex solutions for real (i.e., physical) sources $\rho, \vec{j}$. Then $\mathrm{Re}\,\vec{E}, \mathrm{Re}\,\vec{B}$ are also solutions for $\rho, \vec{j}$.

proof: Corollary I $\rightsquigarrow$ $\mathrm{Re}\,\vec{E}, \mathrm{Re}\,\vec{B}$ are solutions for $\mathrm{Re}\,\rho = \rho$, $\mathrm{Re}\,\vec{j} = \vec{j}$.

remark: (2) In this case, $\mathrm{Im}\,\vec{E}, \mathrm{Im}\,\vec{B}$ are solutions in the absence of sources (since $\mathrm{Im}\,\rho = \mathrm{Im}\,\vec{j} = 0$).

Week 4

hw set 4
(# 12, 13, 14, 15)

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