Chapter 3 Electromagnetic Waves in Vacuum

§1 Plane electromagnetic waves

1.1 The wave equation

Consider vacuum, $j^\mu(x) \equiv 0$, and try to find nonzero solutions of Maxwell's eqs.

remark: (1) Any such solutions must be time dependent: ch 2 §1.1, 1.2 → The static potentials in vacuum obey Laplace's eq., which has only the zero solution.

theorem: In vacuum, and in Lorentz gauge, the 4-vector potential $A^\mu(x)$ obeys

$$\boxed{\partial_\nu \partial^\nu A^\mu(x) = 0} \quad (*)$$

remark: (2) $(*)$ is called 'wave equation'

(3) The operator $\partial_\nu \partial^\nu \equiv \Box$ is called d'Alembert operator. More explicitly it reads

$$\partial_\nu \partial^\nu = g^{\nu\mu}\partial_\nu \partial_\mu = g^{\nu\mu}\frac{\partial^2}{\partial x^\nu \partial x^\mu} = \frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2$$

(3') Some books define $\Box$ as $-\partial_t^2/c^2 + \vec{\nabla}^2$.

proof of theorem: ch 1 §1.3 → In vacuum, $j^\mu = 0$, so we have

$$0 = \partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial^\nu \underbrace{\partial_\mu A^\mu}_{\substack{=0 \text{ in Lorentz} \\ \text{gauge, ch 2 §2.2}}}$$

$$= \partial_\mu \partial^\mu A^\nu \qquad \Box$$

remark: (4) Lorentz gauge implies a Lorentz invariant relation between φ and $\vec{A}$: $0 = \partial_\mu A^\mu = \frac{\partial A^\mu}{\partial x^\mu} = \frac{1}{c}\partial_t \varphi + \vec{\nabla}\cdot\vec{A}$

corollary: The electric and magnetic fields also obey the wave eq.:

$$\boxed{\partial_\mu \partial^\mu \vec{E} = \partial_\mu \partial^\mu \vec{B} = 0} \quad (**)$$

proof: def. 1 § 2.4 $\rightarrow$ $\partial_\mu \partial^\mu \vec{B} = \partial_\mu \partial^\mu \vec{\nabla}\times\vec{A} = \vec{\nabla}\times \partial_\mu\partial^\mu \vec{A} = 0$

$$\partial_\mu \partial^\mu \vec{E} = -\partial_\mu \partial^\mu \vec{\nabla}\varphi - \frac{1}{c}\partial_\mu \partial^\mu \partial_t \vec{A} = 0 \qquad \square$$

remark: (5) Lorentz gauge still does not determine the potentials uniquely, and one can always choose a gauge such that $\varphi = 0$, $\vec{\nabla}\cdot\vec{A} = 0$, see Problem # 8 (NB: this gauge is not Lorentz invariant)

Problem 28

Electromagnetic waves and gauge invariance

1.2 Plane waves

def.: Solutions of the wave eq. that depend on only one spatial coordinate plus time are called plane waves.

Let $f(x,t)$ be any component of $\vec{E}$ or $\vec{B}$. § 1.1 (**) $\rightarrow$

$$\boxed{(\partial_t^2 - c^2\partial_x^2) f = 0} \quad (*) \quad \text{plane-wave equation}$$

theorem: The most general solution of (*) is

$$\boxed{f(x,t) = f_1(x-ct) + f_2(x+ct)}$$

where f_1, f_2 are arbitrary two times cont. differentiable fcts.

3/6/17

remark: (1) PDEs in general have whole classes of fcts as solutions, in contrast to ODEs!

(1') This is called the D'Alembert solution of the wave equation proof: Write (*) as $\left(\frac{1}{c}\partial_t - \partial_x\right)\left(\frac{1}{c}\partial_t + \partial_x\right) f = 0$ (+)

let $\xi := x - ct$, $\eta := x + ct$

$\leadsto x = \frac{1}{2}(\xi + \eta)$, $t = \frac{-1}{2c}(\xi - \eta)$

and $\psi(\xi,\eta) := f(x,t)$

$$\leadsto \frac{1}{c}\partial_t f = (\partial_\xi \psi)\frac{1}{c}\partial_t \xi + (\partial_\eta \psi)\frac{1}{c}\partial_t \eta = -\partial_\xi \psi + \partial_\eta \psi$$

$$\partial_x f = (\partial_\xi \psi)\partial_x \xi + (\partial_\eta \psi)\partial_x \eta = \partial_\xi \psi + \partial_\eta \psi$$

$$(+) \leadsto 0 = -4\partial_\xi \partial_\eta \psi(\xi,\eta)$$

$$\leadsto \partial_\eta \psi(\xi,\eta) = a(\eta, \cancel{\xi})$$

$$\leadsto \psi(\xi,\eta) = \int_{\eta_0}^{\eta} d\tilde{\eta}\, a(\tilde{\eta}) + b(\xi)$$

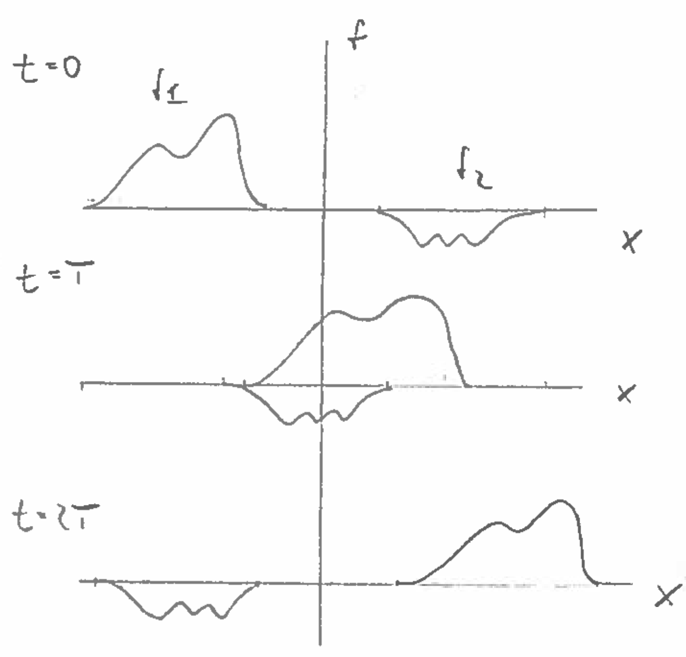
with a and b arbitrary fcts.

let $f_1(\xi) := b(\xi)$, $f_2(\eta) := \int_{\eta_0}^{\eta} d\tilde{\eta}\, a(\tilde{\eta})$

$$\leadsto \psi(\xi,\eta) = f_1(\xi) + f_2(\eta)$$

$$\leadsto \underline{f(x,t) = f_1(x-ct) + f_2(x+ct)} \quad \square$$

remark: (2) f_1 moves in $+x$-direction with velocity c; f_2 moves in $-x$-direction with velocity c; f is superposition of f_1 and f_2.



Problem 29

Plane waves

1.7 The orientation of the fields

<u>proposition</u>: Consider a plane electromagnetic wave that propagates in the direction $\hat{n}$. Then $\vec{E}, \vec{B}$, and $\hat{n}$ are all perpendicular to one another, and $\boxed{\vec{B} = \hat{n} \times \vec{E}}$

<u>proof</u>: Problem 8i $\rightarrow$ We can choose a gauge such that $\varphi = 0, \vec{\nabla}\cdot\vec{A} = 0$

let $\hat{n} = (1,0,0) \rightarrow \vec{A}(\vec{x},t) = \vec{A}(x,t)$

and consider a wave that propagates in the $+x$-direction

$$\rightarrow \vec{A}(\vec{x},t) = \vec{A}(x-ct) = \vec{A}(t-x/c) = \vec{A}(u)$$

where $u := t - x/c$

$$\vec{\nabla}\cdot\vec{A} = 0 \rightarrow \partial_x A_x = 0$$

$$\Box \vec{A} = 0 \rightarrow \partial_t^2 A_x = 0 \rightarrow \partial_t A_x = \text{const.}$$

But $\partial_t A_x = \text{const} \neq 0$ implies $E_x = \text{const} \neq 0$ which does not fall off at ∞ $\rightarrow$ Put $A_x = 0$. ($A_x = \text{const}$ can be made $A_x = 0$ by a gauge transfo)

I don't mean this! Think about this

remark: (1) The wave solution does not fall off either, but its average vanishes.

$$\rightarrow \underline{\vec{A}(u)} = (0, A_y(u), A_z(u)) \; \underline{\perp \hat{n}}$$

$$\rightarrow \underline{\vec{E}} = -\frac{1}{c}\partial_t \vec{A} = -\frac{1}{c}\partial_u \vec{A} \; \underline{\perp \hat{n}}$$

$$\underline{\vec{B}} = \vec{\nabla}\times\vec{A} = (0, -\partial_x A_z, \partial_x A_y) = -\frac{1}{c}\partial_u (0, -A_z, A_y)$$

$$= -\frac{1}{c}\partial_u(\hat{n}\times\vec{A}) = -\frac{1}{c}\hat{n}\times\partial_u\vec{A} = \underline{\hat{n}\times\vec{E}}$$

$\square$

corollary: The Poynting vector is given by

$$\boxed{\vec{P}(\vec{x},t) = c\, u(\vec{x},t)\, \hat{n}}$$

where $u(\vec{x},t)$ is the energy density of the fields.

proof: ch 1 § 3.6 ~> $\vec{P} = \frac{c}{4\pi} \vec{E}\times\vec{B} = \frac{c}{4\pi} \vec{E} \times (\hat{n}\times\vec{E}) \overset{\vec{E}\perp\hat{n}}{=} \frac{c}{4\pi} \hat{n}\, \vec{E}^2$

But $\vec{E}^2 = \vec{B}^2$ ~> $\underline{\underline{\vec{P}}} = \frac{c}{8\pi}(\vec{E}^2+\vec{B}^2)\,\hat{n} = \underline{\underline{c u \hat{n}}}$ □

remark: (2) The energy contained in the wave propagates with velocity c in the direction $\hat{n}$ perpendicular to the wave fronts.

1.4 Monochromatic plane waves

Consider a wave eq.
$$\left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right) f(\vec{x},t) = 0$$

def. 1: A solution of the form $\boxed{f(\vec{x},t) = f_0\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}}$ $f_0 \in \mathbb{C}$

is called a monochromatic plane wave with frequency ω.

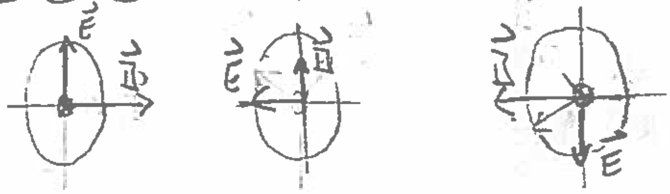
remark: (1) Problem 29 ~> $\omega^2 = c^2 \vec{k}^2$ is necessary and sufficient for f to be a solution.

(2) Superposition principle (ch 1 § 5) ~> If $f \in \mathbb{C}$ is a solution, then so are Re f and Im f.

conv 1: $k_x, k_y, k_z \in \mathbb{R}$ ~> $\omega \in \mathbb{R}$

$f_0 = |f_0| e^{-i\delta}$, $\delta \in \mathbb{R}$

(2') Visualization: for propagation into the plane



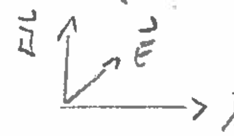
$t=0$ $\quad$ $t=t_1>0$ $\quad$ $t=t_2>t_1$

remark: (1) The direction of propagation $\hat{n}$ from {1.} is

modulus of real vector

$$\hat{n} = \hat{k} = \vec{k}/|\vec{k}| = \vec{k}/(\omega/c)$$

{1.} $\Rightarrow |\vec{E}_0| = |\vec{B}_0|$, and $\vec{E}_0, \vec{B}_0, \hat{k}$ form a right-handed orthogonal system



Consider $\vec{E}$ from now on (analogous considerations hold for $\vec{B}$), and consider $\vec{E}_0$ a complex vector

$\to$ let $\vec{E}_0^{\,2} = |\vec{E}_0^{\,2}| \cdot e^{-2i\alpha}$ complex number with modulus $|\vec{E}_0^{\,2}|$ and phase 2α

$\to \vec{b} = \vec{E}_0 e^{i\alpha}$ (modulus of complex number)

has the property $\vec{b}^{\,2} = |\vec{E}_0^{\,2}| \in \mathbb{R}$

$\to$ consider the physical solution

$$\vec{E}(\vec{x},t) = \mathrm{Re}\left(\vec{b}\, e^{i(\vec{k}\vec{x}-\omega t-\alpha)}\right)$$

where $\vec{b} = \vec{b}_1 + i\vec{b}_2$ with $\vec{b}_1 \perp \vec{b}_2$ (so that $\vec{b}^{\,2} \in \mathbb{R}$)

let $\vec{k} = (k,0,0)$, $\vec{b}_1 = (0,b_1,0)$, $\vec{b}_2 = (0,0,b_2)$

$$\to \quad E_y(\vec{x},t) = b_1 \cos(\vec{k}\vec{x}-\omega t-\alpha)$$

$$E_z(\vec{x},t) = -b_2 \sin(\vec{k}\vec{x}-\omega t-\alpha) \quad \to \quad \boxed{\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1}$$

proposition: The $\vec{E}$-field vector moves on an ellipse.

remark: (2) Monochromatic plane waves are in general elliptically polarized.

(3) Special cases: $b_1 = b_2$ circular polarization

$b_1 = 0$ or $b_2 = 0$ linear polarization

(4) Elliptically polarized wave = superposition of two linearly polarized

remark: (1) & 2 §5.2 $\to$ the obtain physical solution by taking the real part in the end.

1.6 The 4-wave vector, and the Doppler effect

Problem 29 ~> The object $k^\mu = (\omega/c, \vec{k}) \equiv (k_0, \vec{k})$ transforms as a Minkowski vector.

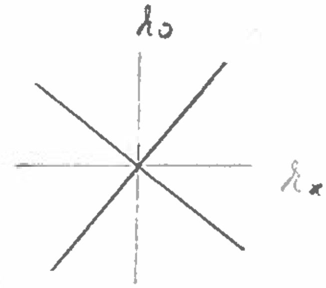
def. 1: k^μ is called 4-wave vector

proposition: The 4-wave vector has zero length in Minkowski space:

$$k_\mu k^\mu = 0$$

proof: $k_\mu k^\mu = k_0^2 - \vec{k}^2 \overset{\text{wave eq.}}{=} 0$ □

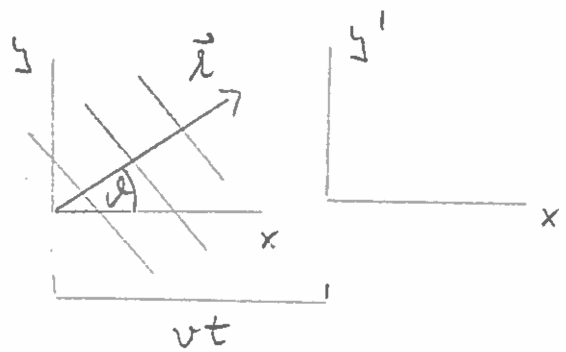
remark: (1) k_μ lies on the light cone given by $k_0 = |\vec{k}|$.



Problem 30

spherical waves

Consider an observer in a moving frame whose velocity forms an angle ϑ with $\vec{k}$.



theorem: If ω is the frequency of the wave in the rest frame, then the observer measures a frequency

$$\boxed{\omega' = \frac{\omega}{\sqrt{1-v^2/c^2}}\left(1-\frac{v}{c}\cos\vartheta\right)}$$

Problem 31

cosmological redshift

proof: Lorentz boost along x-axis (see Problem #29)

$$\rightarrow \quad \omega'/c = \gamma\left(\omega/c - \frac{v}{c}k_x\right)$$

and $k_x = |\vec{k}|\cos\vartheta = \frac{\omega}{c}\cos\vartheta$ □

Week 9

wksht 9 (# 26, 27, 28)

remark: (2) The frequency shift given by $(1-\frac{v}{c}\cos\vartheta)$ is called <u>linear Doppler effect</u>. The one given by $1/\sqrt{1-v^2/c^2}$ is called <u>quadratic Doppler effect</u>.

(3) The quadratic Doppler effect leads to a shift even if $\cos\vartheta = 0$

(4) Consider a nonrelativistic wave, e.g., a sound wave (density wave) in a fluid:

$$\delta n(\vec{x},t) = a\, e^{i(\vec{k}\vec{x}-\omega t)} \qquad \omega = c_0 k \text{ with } c_0 \text{ the phase velocity}$$

Galileo transf.: $x' = x - vt$, $t' = t$, $y' = y$

$$\Rightarrow \delta n(\vec{x},t) = a\, e^{i(k_x x' + k_x v t' + k_y y' - \omega t')} = a\, e^{i(k_x x' + k_y y' - \omega(1 - v\frac{k_x}{\omega})t')}$$

$$\Rightarrow \underline{\omega'} = \omega\left(1 - v\frac{k_x}{\omega}\right) = \underline{\omega\left(1 - \frac{v}{c_0}\cos\vartheta\right)}$$

$\Rightarrow$ Only the linear Doppler effect is observed, <u>no</u> frequency shift if $\vec{v} \perp \vec{k}$!

3/13/17

§2 The wave equation as an initial-value problem

2.1 The wave equation in Fourier space

§1.1 $\Rightarrow$ The general wave eq. reads

$$\boxed{\Box f(\vec{x},t) = \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right) f(\vec{x},t) = 0} \qquad (*)$$

Consider Fourier transf. according to ch. 3 §2:

$$\hat{f}(\vec{k},t) = \int d\vec{x}\, e^{-i\vec{k}\vec{x}} f(\vec{x},t)$$

with Fourier transform

$$f(\vec{x},t) = \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\, \hat{f}(\vec{k},t)$$

remark: (1) Generalised fct. concept → this can be done for a large class of fcts $f(\vec{x},t)$.

$$(*) \to \quad 0 = \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right)\frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\, \hat{f}(\vec{k},t)$$

$$= \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\left[\frac{1}{c^2}\partial_t^2 + \vec{k}^2\right]\hat{f}(\vec{k},t)$$

$$\to \quad \boxed{\frac{d^2}{dt^2}\hat{f}(\vec{k},t) + c^2\vec{k}^2\hat{f}(\vec{k},t) = 0} \quad (**)$$

remark: (2) This is an ODE for a harmonic oscillator with frequency $\omega_{\vec{k}} = c|\vec{k}| \equiv c\,k$!

(3) Fourier back transform known → $(**)$ is equivalent to $(*)$.

2.2 The general solution of the wave equation

General solution of §2.1 $(**)$:

$$\hat{f}(\vec{k},t) = a^0_{\vec{k}}\cos(\omega_{\vec{k}}t) + \frac{\dot{a}^0_{\vec{k}}}{\omega_{\vec{k}}}\sin(\omega_{\vec{k}}t)$$

where

$$\boxed{\begin{aligned} a^0_{\vec{k}} &:= \hat{f}(\vec{k},t=0) = \int d\vec{x}\, e^{-i\vec{k}\vec{x}} f(\vec{x},t=0) \\ \dot{a}^0_{\vec{k}} &:= \frac{d}{dt}\Big|_{t=0}\hat{f}(\vec{k},t) = \int d\vec{x}\, e^{-i\vec{k}\vec{x}}\,\partial_t f(\vec{x},t)\Big|_{t=0} \end{aligned}}$$

known: The general solution of the wave eq. is uniquely determined by the field $f(\vec{x},t=0)$ and its time derivative $\partial_t f(\vec{x},t)|_{t=0}$ at an initial time (wlog $t=0$) and is given by

$$\boxed{f(\vec{x},t) = \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\vec{x}}\left[a^0_{\vec{k}}\cos\omega_{\vec{k}}t + \frac{\dot{a}^0_{\vec{k}}}{\omega_{\vec{k}}}\sin\omega_{\vec{k}}t\right]}$$

with $\omega_{\vec{k}} = c|\vec{k}|$ and $a^0_{\vec{k}}, \dot{a}^0_{\vec{k}}$ the initial values as given above.

corollary: The solution can be written

$$\boxed{f(\vec{x},t) = \frac{1}{(2\pi)^3}\int d\vec{k}\left[f^+_{\vec{k}}\, e^{i(\vec{k}\vec{x}-\omega_{\vec{k}}t)} + f^-_{\vec{k}}\, e^{-i(\vec{k}\vec{x}-\omega_{\vec{k}}t)}\right]}$$

slide 32
general solution
wave eq.

where $f^\pm_{\vec{k}} = \frac{1}{2}\left(a^0_{\pm\vec{k}} \pm i\,\dot{a}^0_{\pm\vec{k}}/\omega_{\vec{k}}\right)$

proof:

$$a^0_{\vec{k}}\cos\omega_{\vec{k}}t + \frac{\dot{a}^0_{\vec{k}}}{\omega_{\vec{k}}}\sin\omega_{\vec{k}}t = a^0_{\vec{k}}\frac{1}{2}\left(e^{i\omega_{\vec{k}}t} + e^{-i\omega_{\vec{k}}t}\right) + \dot{a}^0_{\vec{k}}\frac{1}{2i}\left(e^{i\omega_{\vec{k}}t} - e^{-i\omega_{\vec{k}}t}\right)\frac{1}{\omega_{\vec{k}}}$$

$$= \underbrace{\frac{1}{2}\left(a^0_{\vec{k}} + i\frac{\dot{a}^0_{\vec{k}}}{\omega_{\vec{k}}}\right)}_{}e^{-i\omega_{\vec{k}}t} + \underbrace{\frac{1}{2}\left(a^0_{\vec{k}} - i\frac{\dot{a}^0_{\vec{k}}}{\omega_{\vec{k}}}\right)}_{}e^{i\omega_{\vec{k}}t}$$

$$= f^+_{\vec{k}}\,e^{-i\omega_{\vec{k}}t} + f^-_{-\vec{k}}\,e^{i\omega_{\vec{k}}t}$$

$$\rightarrow \underset{\omega_{-\vec{k}}=\omega_{\vec{k}}}{f(\vec{x},t)} = \frac{1}{(2\pi)^3}\int d\vec{k}\, f^+_{\vec{k}}\, e^{i\vec{k}\vec{x}-\omega_{\vec{k}}t} + \frac{1}{(2\pi)^3}\int d\vec{k}\, f^-_{-\vec{k}}\, e^{i\vec{k}\vec{x}+\omega_{\vec{k}}t}$$

$$\ldots + \frac{1}{(2\pi)^3}\int d\vec{k}\, f^-_{\vec{k}}\, e^{-i\vec{k}\vec{x}+\omega_{\vec{k}}t}$$

remark: (1) The general solution of the wave eq. is a linear superposition of monochromatic plane waves with superposition amplitudes that are uniquely determined by $f(\vec{x},t=0)$ and $\dot{f}(\vec{x},t=0)$.