chapter 4 Electromagnetic Radiation

idea: so far we have discussed

* static solutions of M's eqs with sources (ch 2)
* dynamic solutions in vacuum (ch 3)

Now let's discuss

* dynamic solutions in the presence of sources (charges + currents)

§1 Review of potentials and gauges

1.1 Potentials and fields

ch 1 §2.4 ~> The fields $\vec{E}$ and $\vec{B}$ (which are observable) can be obtained from the potentials φ and $\vec{A}$ (which are not) via

$$\boxed{\begin{aligned}\vec{E}(\vec{x},t) &= -\vec{\nabla}\varphi(\vec{x},t) - \tfrac{1}{c}\partial_t \vec{A}(\vec{x},t)\\ \vec{B}(\vec{x},t) &= \vec{\nabla}\times\vec{A}(\vec{x},t)\end{aligned}}$$

remark: (1) The first two Maxwell eqs (i.e., the homogeneous ones) are then automatically fulfilled.

(2) ch 1 §3.1 ~> $\varphi, \vec{A}$ are the components of a 4-vector

$$A^\mu(x) = (\varphi(x), \vec{A}(x))$$

proposition: The inhomogeneous M eqs (i.e., the 3rd and 4th) are equivalent to 4 PDEs for $A^\nu(x)$:

$$\boxed{\partial_\mu \partial^\mu A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = \frac{4\pi}{c} j^\nu(x)} \qquad (*)$$

proof: ch 1 §1.3 ~> $\frac{4\pi}{c} j^\nu = \partial_\mu F^{\mu\nu} \overset{\text{ch 1 §1.1}}{=} \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu$

corollary: In terms of φ and $\vec{A}$, $(*)$ reads

$$\boxed{\begin{array}{ll} \Box \vec{A} + \vec{\nabla}\left(\frac{1}{c}\partial_t \varphi + \vec{\nabla}\cdot\vec{A}\right) = \frac{4\pi}{c}\vec{j} \\ \hline -\vec{\nabla}^2 \varphi - \frac{1}{c}\partial_t \vec{\nabla}\cdot\vec{A} = 4\pi \rho \end{array}} \qquad (*')$$

where $\quad \Box = \frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2$

proof: $j^\nu = (c\rho, \vec{j})$, $\quad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\partial_t, -\vec{\nabla}\right)$

$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t, \vec{\nabla}\right)$

$\rightarrow \partial_\mu \partial^\mu = \frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2 = \Box$, and $\partial_\mu A^\mu = \frac{1}{c}\partial_t \varphi + \vec{\nabla}\cdot\vec{A}$ $\Box$

remark: (3) In the static case, $(*')$ simplifies to

$$\vec{\nabla}^2 \varphi = -4\pi\rho \qquad \text{Poisson's eq, cf 2 §1.1 ✓}$$

$$\underbrace{-\vec{\nabla}^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})}_{= \vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{B}} = \frac{4\pi}{c}\vec{j} \qquad \text{4th M eq ✓}$$

(4) In vacuum, $(*)$ simplifies to

$$\Box A^\nu(x) - \partial^\nu \partial_\mu A^\mu(x) = 0$$

which in Lorenz gauge ($\partial_\mu A^\mu = 0$, see below) further simplifies to the wave eq

$$\Box A^\nu(x) = 0 \qquad \text{cf 3 §1.1 ✓}$$

1.2 Gauge invariance

cf 1 §2.4 → the potentials are not unique

→ We can choose certain constraints on the potentials ("choosing a gauge").

remark: (J') This corollary is logically necessary: consider the space of all fields A^r, $\{A^r\}$. Now fix the gauge. This restricts us to a subspace $\{A^r\}_g$ of $\{A^r\}$, and we know it suffices to consider that subspace. It then is important to ascertain that the equation of motion does not take us out of the subspace, and that's what the corollary does

Popular choices are

(1) <u>Lorenz gauge</u> $\boxed{\partial_\mu A^\mu(x) = 0}$ or $\boxed{\frac{1}{c}\partial_t \varphi(\vec{x},t) + \vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0}$

(2) <u>Coulomb gauge</u> $\boxed{\vec{\nabla}\cdot\vec{A}(x) = 0}$ cf. Problem 8

remark: (1) Some books call this "transverse gauge", and some call it "radiation gauge".

(2) Another possibility is to choose $\varphi(x) = 0$. This also sometimes is called "radiation gauge".

(3) 4 potentials + 1 constraint $\rightarrow$ 3 fields determine the 6 observable fields $\vec{E}, \vec{B}$.

proposition 1: In Lorenz gauge the eq. of motion for the potentials, §1.1 (*'), becomes

Problem 11
Wave eqs for fields

$$\boxed{\begin{array}{l} \Box \vec{A} = \frac{4\pi}{c}\vec{j} \\ \Box \varphi = 4\pi\rho \end{array}} \quad \text{or} \quad \boxed{\Box A^\mu = \frac{4\pi}{c} j^\mu} \qquad (*)$$

proof: Lorenz gauge $\rightarrow \vec{\nabla}\cdot\vec{A} = -\frac{1}{c}\partial_t\varphi$

$\rightarrow -\frac{1}{c}\partial_t \vec{\nabla}\cdot\vec{A} = \frac{1}{c^2}\partial_t^2\varphi$

$\rightarrow$ (*) follows immediately from §1.1 (*').

corollary 1: Once we choose Lorenz gauge, it is maintained under time evolution.

proof: $\underline{\Box\,\partial_\mu A^\mu} \overset{(*)}{=} \frac{4\pi}{c}\partial_\mu j^\mu = \underline{0}$ by charge conservation, ch. 1 §2.1

remark: (4) ch. 1 §2.1 $\rightarrow \partial_\mu j^\mu = 0$ is not an independent condition, but follows from the field eqs.

Problem 34 Potentials

proposition 2: In Coulomb gauge the eqs of motion become

$$\boxed{\begin{array}{l} \Box \vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\partial_t \vec{\nabla}\varphi \\ \hline \vec{\nabla}^2\varphi = -4\pi\rho \end{array}} \qquad (**)$$

proof: §1.1 (*') with $\vec{\nabla}\cdot\vec{A} = 0 \rightarrow (**)$

remark: (5) The eq. for φ is now the same as in electrostatics (see ch 2 §3.1), but φ and ρ are both time dependent

corollary 2: Once we choose Coulomb gauge, it is maintained under time evolution.

proof:
$$\Box\, \vec{\nabla}\cdot\vec{A} \overset{(**)(1)}{=} \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j} - \frac{1}{c}\partial_t \vec{\nabla}^2\varphi \overset{(**)(2)}{=} \frac{4\pi}{c}\vec{\nabla}\cdot\vec{j} + \frac{4\pi}{c}\partial_t\rho$$
$$= \frac{4\pi}{c}\partial_\mu j^\mu = 0 \quad \text{by charge conservation}$$

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remark: (6) Which gauge to pick is a matter of choice. Different choices are more or less convenient for different applications (although they are all equivalent)

§2 Green's functions in the Lorentz gauge

2.1 The concept of a Green's function

Consider an inhomogeneous wave eq.

$$\boxed{\Box f(\vec{x},t) = i(\vec{x},t)} \qquad (*)$$

with $i(\vec{x},t)$ a given inhomogeneity.

def. 1: A Green's function $G(\vec{x},t)$ for the eq. (*) is a solution of

$$\boxed{\Box G(\vec{x},t) = \delta(\vec{x})\,\delta(t)} \qquad (**)$$

remark: This is the wave eq. (*) with a special inhomogeneity $i(\vec{x},t) = \delta(\vec{x})\delta(t)$
$= \delta(x)\delta(y)\delta(z)\delta(t)$

proposition: let $G(\vec{x},t)$ be a solution of (**). Then

$$\boxed{f(\vec{x},t) = \int d\vec{x}'dt'\, G(\vec{x}-\vec{x}', t-t')\, i(\vec{x}',t')}$$

is a solution of (*).

proof: $\Box f(\vec{x},t) = \int d\vec{x}'dt'\, \Box G(\vec{x}-\vec{x}', t-t')\, i(\vec{x}',t')$

$$= \int d\vec{x}'dt'\, \delta(\vec{x}-\vec{x}', t-t')\, i(\vec{x}',t') = i(\vec{x},t).$$

2.2 Green's fcts for the wave equation

Consider a Fourier trafo of §2.1 (**) with respect to t:

$$\int dt\, e^{i\omega t}\, G(\vec{x},t) =: G_\omega(\vec{x})$$

$$\rightsquigarrow \quad \delta(\vec{x}) \underbrace{\int dt\, e^{i\omega t}\, \delta(t)}_{=1} = \int dt\, e^{i\omega t}\, \frac{1}{c^2}\partial_t^2 G(\vec{x},t) - \vec{\nabla}^2 G_\omega(\vec{x})$$

$$\overset{\text{part. int}}{=} -\frac{\omega^2}{c^2} G_\omega(\vec{x}) - \vec{\nabla}^2 G_\omega(\vec{x})$$

$\rightsquigarrow$ $G_\omega(\vec{x})$ obeys $\boxed{-\left(\vec{\nabla}^2 + \frac{\omega^2}{c^2}\right) G_\omega(\vec{x}) = \delta(\vec{x})}$

solution by Fourier trafo: $G_\omega(\vec{k}) := \int d\vec{x}\, e^{-i\vec{k}\vec{x}}\, G_\omega(\vec{x})$

$$\rightsquigarrow \quad \left(\vec{k}^2 - \omega^2/c^2\right) G_\omega(\vec{k}) = 1$$

$$\rightsquigarrow \quad \boxed{G_\omega(\vec{k}) = \frac{1}{\vec{k}^2 - \omega^2/c^2}}$$

Now Fourier back transform: PHYS 610 $\rightsquigarrow$ $\int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\vec{x}}\, \frac{4\pi}{\vec{k}^2 + 1/r_0^2} = \frac{e^{-r/r_0}}{r}$

$$\rightsquigarrow \quad G_\omega(\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\vec{x}}\, \frac{1}{\vec{k}^2 + (i\omega/c)^2} = \frac{1}{4\pi}\,\frac{1}{r}\, e^{\pm i\omega r/c}$$ where $r = |\vec{x}|$

and Fourier backtransform with respect to t:

$$\underline{G(\vec{x},t)} = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, G_\omega(\vec{x}) = \frac{1}{4\pi r}\,\frac{1}{2\pi}\int d\omega\; e^{-i\omega t \pm i\omega r/c}$$

$$= \frac{1}{4\pi r}\,\frac{1}{2\pi}\int d\omega\; e^{-i\omega(t \mp r/c)} \overset{6.10}{=} \underline{\frac{1}{4\pi r}\,\delta(t \mp r/c)}$$

theorem: The defining eq. for the Green's fct, §2.1 (t.c), has two solutions $\boxed{G_\pm(\vec{x},t) = \frac{1}{4\pi r}\,\delta(t \mp r/c)}$ $\quad r = |\vec{x}|$

remark: (1) Consider a time-dependent point source, $i(\vec{x},t) = i(t)\delta(\vec{x})$

§2.1 prop. ⇝ The two solutions of the inhomogeneous wave eq. given by $G_\pm$ are

$$\underline{f_\pm(\vec{x},t)} = \int d\vec{x}'\,dt'\,\frac{1}{4\pi|\vec{x}-\vec{x}'|}\,\delta\!\left(t-t' \mp \frac{1}{c}|\vec{x}-\vec{x}'|\right) i(t')\,\delta(\vec{x}')$$

$$= \frac{1}{4\pi r}\int dt'\,\delta\!\left(t \mp \frac{r}{c} - t'\right) i(t')$$

$$= \underline{\frac{1}{4\pi r}\, i(t \mp r/c)}$$

⇝ If the source $i(t')$ is active at time t', then the field response occurs at a time $t = t' \pm r/c$ for the solutions $f_\pm$.

def: G_+ is called retarded Green's fct., G_- is called advanced Green's fct.

axiom: Causality

A physical response cannot precede the action of the source.

consequence: Only the retarded solution is physical!

remark: (2) Advanced Green's fcts are nonetheless useful in QFT, both in high-energy physics and in statistical mechanics

2.3 The retarded potentials

Return to the wave eqs for $\vec{A}$ and φ, §1.2.

§2.1 proposition + §2.2 ->

$$\varphi(\vec{x},t) = \int d\vec{x}'dt' \frac{1}{4\pi|\vec{x}-\vec{x}'|} \delta(t-t'-|\vec{x}-\vec{x}'|/c)\, 4\pi\rho(\vec{x}',t')$$

$$\to \boxed{\varphi(\vec{x},t) = \int d\vec{y} \frac{1}{|\vec{x}-\vec{y}|} \rho(\vec{y},t-|\vec{x}-\vec{y}|/c)} \qquad (*)$$

Analogously,

$$\boxed{\vec{A}(\vec{x},t) = \frac{1}{c}\int d\vec{y} \frac{1}{|\vec{x}-\vec{y}|} \vec{j}(\vec{y},t-|\vec{x}-\vec{y}|/c)} \qquad (**)$$

remark: (1) (*), (**) are called retarded potentials

(2) The time delay $\Delta t = |\vec{x}-\vec{y}|/c$ corresponds to the time it takes a signal to travel from point $\vec{y}$ to point $\vec{x}$ with velocity c.

Problem 35

Liénard-Wiechert potentials

(3) (*), (**) are analogous to Poisson's formula in the static case, cf ch 2 §2.3, 2.6. New concept introduced by dynamical sources: Finite propagation velocity causes retardation.

Problem 36

Potential of a moving charge

Week 1

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End Winter '18

remark: (2') Note the difference between $\left(\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t)\right)_{t=t_r}$, where the $\vec{\nabla}_\xi$ acts only on the explicit $\vec{\xi}$-dependence of $\vec{j}$, and $\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t_r)$, where it acts both on the explicit $\vec{\xi}$-dependence and the implicit one hidden in t_r, which is $\vec{\xi}$-dependent!

lemma 2: $\boxed{\partial_t \rho(\vec{\xi},t_r) = -\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t_r) + \frac{1}{c}\hat{x}\cdot\partial_t \vec{j}(\vec{\xi},t_r)}$

proof: continuity eq. (ch 1 §2.1): $\partial_t \rho(\vec{x},t) = -\vec{\nabla}_x \cdot \vec{j}(\vec{x},t)$

$$\rightarrow \quad \partial_t \rho(\vec{\xi},t_r) = -\left(\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t)\right)_{t=t_r}$$

but $\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t_r) = \left(\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t)\right)_{t=t_r} + \partial_t \vec{j}(\vec{\xi},t_r)\cdot \underbrace{\vec{\nabla}_\xi t_r}_{=\frac{1}{c}\hat{x}}$

$$= \left(\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t)\right)_{t=t_r} + \frac{1}{c}\hat{x}\cdot\partial_t \vec{j}(\vec{\xi},t_r)$$

$$\rightarrow \quad \underline{\partial_t \rho(\vec{\xi},t_r)} = \underline{-\vec{\nabla}_\xi \cdot \vec{j}(\vec{\xi},t_r) + \frac{1}{c}\hat{x}\cdot\partial_t \vec{j}(\vec{\xi},t_r)} \qquad \square$$

§3 Radiation by time dependent sources

3.1 Asymptotic potentials and fields

Consider the retarded potentials, §2.3 (*), (**), at large distances $r = |\vec{x}|$ from the charge and current distributions:

$$|\vec{x}-\vec{y}| = \sqrt{r^2 - 2\vec{x}\cdot\vec{y} + y^2} = r\sqrt{1-2\hat{x}\cdot\vec{y}/r + O(1/r^2)} = r - \hat{x}\cdot\vec{y} + O(1/r)$$

$$\rightarrow \boxed{\begin{aligned} \varphi(\vec{x},t) &= \frac{1}{r}\int d\vec{y}\; \rho(\vec{y}, t_r) + O(1/r^2) \\ \vec{A}(\vec{x},t) &= \frac{1}{r}\frac{1}{c}\int d\vec{y}\; \vec{j}(\vec{y}, t_r) + O(1/r^2) \end{aligned}}$$

where $t_r := t - \frac{r}{c} + \frac{1}{c}\hat{x}\cdot\vec{y}$

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remark: (1) We keep only the leading contribution for $r\to\infty$, which is of $O(1/r)$

(2) How many terms to keep in the time argument t_r of ρ and $\vec{j}$ depends on how rapidly the sources are changing. If L is the linear extension of the source, and the source changes appreciably on a time scale $\Delta t = L/c$, then the term $\frac{1}{c}\hat{x}\vec{y}$ in the time argument will be important.

lemma 1: $\boxed{\vec{\nabla}\frac{1}{r}f(t_r) = -\frac{1}{c}\hat{x}\frac{1}{r}\partial_t f(t_r) + O(1/r^2)}$

proof: $\vec{\nabla}\frac{1}{r}f(t_r) = \left(\vec{\nabla}\frac{1}{r}\right)f(t_r) + \frac{1}{r}\partial_t f(t_r)\,\vec{\nabla}t_r$

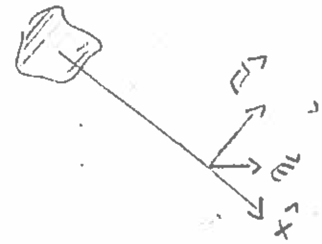
$$= O(1/r^2) + \frac{1}{r}\partial_t f(t_r)\left(\frac{-1}{c}\right)\vec{\nabla}\sqrt{x^2+y^2+z^2}$$

$$= \frac{-1}{c}\frac{1}{r}\partial_t f(t_r)\frac{\vec{x}}{r} = -\frac{1}{c}\frac{\vec{x}}{r}\partial_t f(t_r) + O(1/r^2)$$

□

proposition: For far from the sources, the fields are given by

$$\boxed{\begin{aligned}\vec{B}(\vec{x},t) &\simeq -\frac{1}{c^2}\frac{\hat{x}}{r}\times\int d^3y\ \partial_t \vec{j}(\vec{y},t_r)\\ \vec{E}(\vec{x},t) &\simeq -\ \hat{x}\times\vec{B}(\vec{x},t)\end{aligned}}$$



remark: (3) This implies $\vec{E}^2=\vec{B}^2$, and $\hat{x}\perp\vec{E}\perp\vec{B}$ form a right-handed orthogonal system

(4) The fields fall off only as $1/r$, as opposed to $1/r^2$ in static solutions of M's eqs!

Problem 37 solutions in related gauge

(4') These results are independent of the gauge chosen, see Problem 37

proof of proposition: §1.1 $\Rightarrow$ $\vec{B}=\nabla\times\vec{A}$

$$\Rightarrow B_i = \varepsilon_{ijk}\,\partial_j\,\frac{1}{r}\frac{1}{c}\int d^3y\ j_k(\vec{y},t_r) \overset{\text{lemma 1}}{=} \varepsilon_{ijk}\,\frac{-1}{c^2}\frac{\hat{x}_j}{r}\int d^3y\ \partial_t j_k(\vec{y},t_r)$$

$$\Rightarrow \underline{\underline{\vec{B}}} = \underline{\underline{-\frac{1}{c^2}\frac{\hat{x}}{r}\times\int d^3y\ \partial_t\vec{j}(\vec{y},t_r)}}$$

and $$\underline{\underline{\vec{E}}} = -\nabla\varphi-\frac{1}{c}\partial_t\vec{A} \overset{\text{lemma 1}}{=} +\frac{1}{c}\frac{\hat{x}}{r}\int d^3y\ \partial_t\rho(\vec{y},t_r)-\frac{1}{c}\frac{1}{r}\frac{1}{c}\int d^3y\ \partial_t\vec{j}(\vec{y},t_r)$$

$$\overset{\text{lemma (2)}}{=} -\frac{1}{c}\frac{\hat{x}}{r}\underbrace{\int d^3y\ \nabla_y\cdot\vec{j}(\vec{y},t_r)}_{=\oint_{\partial V} d\vec{S}\cdot\vec{j}(\vec{y},t_r)\,=\,0}+\frac{1}{c}\frac{\hat{x}}{r}\int d^3y\ \frac{\hat{x}}{c}\cdot\partial_t\vec{j}(\vec{y},t_r)-\frac{1}{c^2 r}\int d^3y\ \partial_t\vec{j}(\vec{y},t_r)$$

$$=\frac{1}{c^2 r}\int d^3y\left[\hat{x}\left(\hat{x}\cdot\partial_t\vec{j}(\vec{y},t_r)\right)-\partial_t\vec{j}(\vec{y},t_r)\right]$$

$$\left(\vec{a}\times(\vec{a}\times\vec{b})\right)_i = \varepsilon_{ijk}\,a_j\,\varepsilon_{klm}\,a_l b_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\,a_j a_l b_m = a_i(\vec{a}\cdot\vec{b})-(\vec{a})^2 b_i$$

$\vec{a}=\hat{x}$, $\vec{b}=\partial_t\vec{j}$

$$=\frac{1}{c^2 r}\int d^3y\ \hat{x}\times\left(\hat{x}\times\partial_t\vec{j}(\vec{y},t_r)\right)=\hat{x}\times\left(\frac{1}{c^2 r}\,\hat{x}\times\int d^3y\ \partial_t\vec{j}(\vec{y},t_r)\right)$$

$$=-\ \hat{x}\times\vec{B}$$

$\square$

remark: (5) A time-dependent localized current density leads to time-dependent fields everywhere in space (with proper retardation for the t-dependence of the fields). This phenomenon is called radiation

(6) Far from the source, the radiation fields $\vec{E}$ and $\vec{B}$

(i) fall off as $1/r$

(ii) are perpendicular to one another and perpendicular to the radius vector from the source to the observer.

(7) The source must provide the field energy $\rightarrow$ steady power loss of the source!

4/10/17

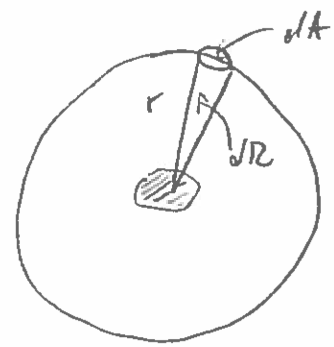
3.2 The radiated power

cf 1 § 3.6 $\rightarrow$ the energy-current density of the fields is given by the Poynting vector: $\vec{P}(\vec{x},t) = \frac{c}{4\pi}\,\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)$

remark: (1) $\vec{E}\perp\vec{B}\perp\hat{x} \rightarrow \vec{P}\parallel\hat{x}$

(2) $[\vec{P}]$ = energy per unit area and unit time

(3) $\hat{x}\cdot\vec{P}$ = power per unit area



area element $dA = r^2 d\Omega$

with solid-angle element $d\Omega$

$\rightarrow$ the radiated power $\mathcal{P}$ per solid angle is

$$\frac{d\mathcal{P}}{d\Omega} = r^2\,\hat{x}\cdot\vec{P} = r^2\frac{c}{4\pi}\,\hat{x}\cdot(\vec{E}\times\vec{B})$$

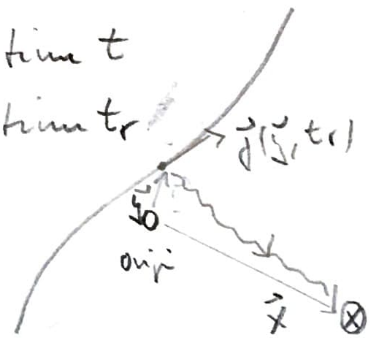
$$= -r^2\frac{c}{4\pi}\,\hat{x}\cdot\left((\hat{x}\times\vec{B})\times\vec{B}\right) = r^2\frac{c}{4\pi}\,\vec{B}^2$$

75/f.p.

§2.1 ⇝ We need $\vec{j}(\vec{\xi},t)$ at the retarded time

signal received: at point $\vec{x}$ and time t

signal emitted: at point $\vec{\xi}$ and time t_r



§2.3 ⇝ $t_r = t - \frac{1}{c}|\vec{x}-\vec{\xi}|$

but $\vec{\xi} = \vec{R}(t)$ is t-dependent!

⇝ $\boxed{t_r = t - \frac{1}{c}|\vec{x}-\vec{R}(t_r)|}$ implicit eq. for t_r

$\overset{§2.1}{\simeq} t - \frac{r}{c} + \frac{1}{c}\hat{x}\cdot\vec{R}(t_r) + O(1/r)$

$\simeq t - \frac{r}{c}$ for $v \ll c$

$\underline{=: t_e}$

$$= \frac{c}{4\pi} r^2 \frac{1}{c^4 r^2} \left[\hat{x} \times \int d^3y \, \partial_t \vec{j}(\vec{y}, t_r) \right]^2$$

$$= \frac{1}{4\pi c^3} \left(\hat{x} \times \int d^3y \, \partial_t \vec{j}(\vec{y}, t_r) \right)^2$$

theorem: The power radiated by the source per solid angle is

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left(\hat{x} \times \int d^3y \, \partial_t \vec{j}(\vec{y}, t_r) \right)^2}$$

remark: (4) Power $\propto$ (fields)^2 and fields $\propto 1/r$
-> nonzero power per solid angle we observe far away from the source!

corollary: The total power radiated is $\boxed{P = \int d\Omega \, dP/d\Omega}$

3.3 Radiation by an accelerated charged point particle

Consider a point particle with charge e that moves with velocity $v \ll c$ on a trajectory $\vec{R}(t)$. -> current density

$$\vec{j}(\vec{y}, t) = e \vec{v}(t) \, \delta(\vec{y} - \vec{R}(t))$$

remark: (0) The well defined "time of emission" t_e is an approximate expression for the retarded time t_r that's valid for $v \ll c$.

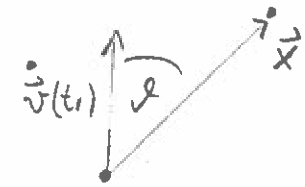
-> $$\int d^3y \, \partial_t \vec{j}(\vec{y}, t_r) \simeq \frac{d}{dt} \int d^3y \, \vec{j}(\vec{y}, t_e) = \frac{d}{dt} e \int d^3y \, \vec{v}(t_e) \, \delta(\vec{y} - \vec{R}(t_e))$$

$$= e \left. \frac{d\vec{v}}{dt} \right|_{t=t_e} = e \dot{\vec{v}}(t_e)$$

consider $\quad (\hat{x}\times\dot{\vec{v}})^2 = \varepsilon_{ijk}\,\hat{x}_j\dot{v}_k\,\varepsilon_{ilm}\,\hat{x}_l\dot{v}_m =$

$$= (\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})\,\hat{x}_j\hat{x}_l\dot{v}_k\dot{v}_m = (\dot{\vec{v}})^2-(\hat{x}\cdot\dot{\vec{v}})^2$$

$$\to \quad \frac{d\mathcal{P}}{d\Omega} = \frac{1}{4\pi c^3}\,e^2\left[\dot{\vec{v}}^2(t_r)-(\hat{x}\cdot\dot{\vec{v}}(t_r))^2\right]$$



Let ϑ be the angle between the acceleration at time t_r and the radius vector to the observer

$$\to \quad (\hat{x}\cdot\dot{\vec{v}})^2 = (\dot{\vec{v}})^2\cos^2\vartheta$$

$$\to \quad \boxed{\frac{d\mathcal{P}}{d\Omega} = \frac{e^2}{4\pi c^3}\left(\dot{\vec{v}}(t_r)\right)^2\sin^2\vartheta}$$

Problem 38

Radiation from cyclotron motion

Problem 39

Radiating harm. osc.

Problem 40

classical atom

proposition: The power radiated by the accelerated charge is

$$\boxed{\mathcal{P} = \frac{2e^2}{3c^3}(\dot{\vec{v}})^2} \qquad (\text{for } v \ll c)$$

proof: $\int d\Omega\,\sin^2\vartheta = 2\pi\int_{-1}^{1}d\eta\,(1-\eta^2) = 4\pi\left(1-\frac{1}{3}\right) = \frac{8\pi}{3}$

$$\to \quad \underline{\mathcal{P}} = \int d\Omega\; d\mathcal{P}/d\Omega = \frac{8\pi}{3}\frac{e^2}{4\pi c^3}(\dot{\vec{v}})^2 = \underline{\underline{\frac{2e^2}{3c^3}(\dot{\vec{v}})^2}} \qquad \square$$

remark: (1) This result is sometimes called the Larmor formula. It is valid for nonrelativistic particles.

(2) This is the physics behind synchrotron radiation, see Problem 38

(3) It implies that a classical atom (classical electron in bounded motion around a proton) cannot be stable, see Problems 38, 39.

3.4 Dipole radiation

Now consider a system of many slow moving ($v \ll c$) charges that is still small compared to r → $t_r \approx t_e$

proposition: In this case the radiated power per solid angle is

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left(\hat{x} \times \ddot{\vec{d}}\right)^2} \quad (\text{for } v \ll c)$$

(0) With θ the angle between $\vec{d}$ and $\hat{x}$ we have $\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \sin^2\theta \, (\ddot{\vec{d}})^2$

where $\vec{d}$ is the dipole moment of the charge distribution

$$\vec{d}(t) = \int d\vec{y}\; \vec{y}\, \rho(\vec{y}, t)$$

and $\ddot{\vec{d}}$ is its second time derivative

remark: (1) For a point charge, $\rho(\vec{y}, t) = e\, \delta(\vec{y} - \vec{R}(t))$

$$\to \vec{d}(t) = e \int d\vec{y}\; \vec{y}\, \delta(\vec{y} - \vec{R}(t)) = e \vec{R}(t)$$

$$\to \ddot{\vec{d}} = e \frac{d}{dt} \vec{v}(t) = e \dot{\vec{v}}$$

→ We recover the proposition from §3.3.

lemma: $$\boxed{\frac{d}{dt} \vec{d}(t) = \int d\vec{y}\; \vec{j}(\vec{y}, t)}$$

proof: charge conservation → $\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$

$$\to 0 = \int d\vec{y}\; \vec{y} \left[\vec{\nabla}_y \cdot \vec{j}(\vec{y}, t) + \partial_t \rho(\vec{y}, t)\right]$$

$$= \int d\vec{y} \left[\underbrace{\vec{\nabla}_y (\vec{y} \cdot \vec{j})}_{=0 \text{ if } \vec{j} \text{ falls off fast enough at } \infty} - \vec{j} + \vec{y}\, \partial_t \rho\right]$$

$$= -\int d\vec{y}\; \vec{j}(\vec{y}, t) + \frac{d}{dt} \int d\vec{y}\; \vec{y}\, \rho(\vec{y}, t)$$

$$= -\int d\vec{y}\; \vec{j}(\vec{y}, t) + \frac{d}{dt} \vec{d}(t) \quad \blacksquare$$

proof of proposition: §3.2 →

$$4\pi c^3 \frac{dP}{d\Omega} \approx \left(\hat{x}\times\int d\vec{y}\ \partial_t \vec{j}(\vec{y},t_e)\right)^2 = \left(\hat{x}\times\frac{d}{dt}\int d\vec{y}\ \vec{j}(\vec{y},t)\right)^2 \overset{\text{lem}}{=} \underline{\underline{\left(\hat{x}\times\ddot{\vec{d}}(t)\right)^2}}$$

remark: (2) This contribution to the radiation field is called <u>electric dipole radiation</u>

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Now consider corrections to the approximation $t_r \approx t_e$.

§3.2 → $dP/d\Omega$ is determined by

$$\underline{\int d\vec{y}\ \ \vec{j}(\vec{y},t_r)} = \int d\vec{y}\ \vec{j}(\vec{y},\underbrace{t-\tfrac{r}{c}}_{=t_e}+\tfrac{1}{c}\hat{x}\cdot\vec{y}+\ldots)$$

$$= \int d\vec{y}\ \vec{j}(\vec{y},t_e) + \frac{1}{c}\int d\vec{y}\ (\hat{x}\cdot\vec{y})\ \partial_t\vec{j}(\vec{y},t)\Big|_{t=t_e} + \ldots$$

$$= \dot{\vec{d}}(t_e) + \frac{1}{c}\frac{d}{dt}\Big|_{t_e}\int d\vec{y}\ \Big[\tfrac{1}{2}(\hat{x}\cdot\vec{y})\vec{j} + \tfrac{1}{2}(\hat{x}\cdot\vec{j})\vec{y} + \tfrac{1}{2}(\hat{x}\cdot\vec{y})\vec{j} - \tfrac{1}{2}(\hat{x}\cdot\vec{j})\vec{y}\Big]$$

$$= \dot{\vec{d}}(t_e) - \frac{1}{2c}\frac{d}{dt}\Big|_{t_e}\int d\vec{y}\ \big[\vec{y}(\hat{x}\cdot\vec{j}) - \vec{j}(\hat{x}\cdot\vec{y})\big] + \text{another term}$$

$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$\curvearrowright = \dot{\vec{d}}(t_e) - \frac{1}{2c}\frac{d}{dt}\Big|_{t_e}\int d\vec{y}\ \hat{x}\times(\vec{y}\times\vec{j}) + \text{another term}$$

$$= \dot{\vec{d}}(t_e) - \hat{x}\times\frac{d}{dt}\Big|_{t_e}\frac{1}{2c}\int d\vec{y}\ \vec{y}\times\vec{j}(\vec{y},t) + \text{another term}$$

$$\underline{= \dot{\vec{d}}(t_e) - \hat{x}\times\dot{\vec{m}}(t_e)} + \text{another term}$$

with $\vec{m}(t) = \frac{1}{2c}\int d\vec{y}\ \vec{y}\times\vec{j}(\vec{y},t)$ the magnetic dipole moment

from ch 2 §3.7

remark: (3) This is the obvious continuum generalization of $\vec{m}$

as defined in ch 2 §5.7

→ In this approximation the power per solid angle is

$$\boxed{\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}\left[\hat{x}\times\left(\ddot{\vec{d}} - \hat{x}\times\ddot{\vec{m}}\right)\right]^2}$$

with $\vec{d}$ and $\vec{m}$ the electric and magnetic dipole moments of the source.

corollary: The total radiated power is

$$\boxed{P = \frac{2}{3c^3}\left[(\ddot{\vec{d}})^2 + (\ddot{\vec{m}})^2\right]}$$

Problem 41
Sum of pole radiation

proof: $\int d\Omega\,(\hat{x}\times\vec{a})^2 = 2\pi\int_{-1}^{1} d\xi\,(1-\xi^2)\,\vec{a}^2 = 4\pi(1-\frac{1}{3})\vec{a}^2 = \frac{8\pi}{3}\vec{a}^2$

$\int d\Omega\,(\hat{x}\times(\hat{x}\times\vec{a}))^2 = \int d\Omega\,(\hat{x}(\hat{x}\cdot\vec{a}) - \vec{a})^2$

$= \int d\Omega\,[\xi^2\vec{a}^2 - 2\xi^2\vec{a}^2 + \vec{a}^2] = 2\pi\int_{-1}^{1} d\xi\,(1-\xi^2)\,\vec{a}^2 = \frac{8\pi}{3}\vec{a}^2$

$\int d\Omega\,(\hat{x}\times\vec{a})\cdot(\hat{x}\times(\hat{x}\times\vec{b})) = 0$ since it's linear in $\hat{x}$

□

remark: (4) The "other term" has the structure

$$\int d\vec{y}\,(y_i j_j + y_j j_i) \overset{\text{part. int}}{=} -\int d\vec{y}\; y_i y_j\, \vec{\nabla}_y\cdot\vec{j} \overset{\text{cont. eq.}}{=} \int d\vec{y}\; y_i y_j\, \partial_t \rho$$

$$= \frac{d}{dt}\int d\vec{y}\; y_i y_j\, \rho(\vec{y},t) = \frac{d}{dt} Q_{ij}(t)$$

Problem 42
Rotating dipole

with Q_{ij} the quadrupole moment

→ the contribution to P from this term is of $O(\frac{1}{c^5}\dddot{Q}^2)$

(5) The magnetic dipole moment has a $1/c$ in its definition
→ magnetic dipole and electric quadrupole radiation are of the same order in v/c and should really be considered together, see LL §71.

§4 Spectral distribution of radiated energy

In §3 we calculated the total power radiated by a time-dependent source.

Question: How is this energy distributed over different frequencies?

4.1 Retarded potentials in frequency space

§2.3 $\to$ $\varphi(\vec{x},t) = \int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|}\, \rho(\vec{y}, t - |\vec{x}-\vec{y}|/c)$

Define a temporal Fourier transform (cf. §2.2)

$$f(\vec{x},\omega) := \int dt\, e^{i\omega t} f(\vec{x},t)$$

$$f(\vec{x},t) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} f(\vec{x},\omega)$$

$$\to \underline{\varphi(\vec{x},\omega)} = \int dt\, e^{i\omega t} \int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|} \int \frac{d\omega'}{2\pi}\, e^{-i\omega'(t-\frac{1}{c}|\vec{x}-\vec{y}|)}\, \rho(\vec{y},\omega')$$

$$= \int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|} \int \frac{d\omega'}{2\pi}\, \rho(\vec{y},\omega')\, e^{i\omega'|\vec{x}-\vec{y}|/c} \underbrace{\int dt\, e^{i(\omega-\omega')t}}_{=2\pi\delta(\omega-\omega')}$$

$$= \int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|}\, e^{i\omega|\vec{x}-\vec{y}|/c}\, \rho(\vec{y},\omega)$$

proposition: The retarded potentials in frequency space are

$$\boxed{\varphi(\vec{x},\omega) = \int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|}\, e^{i\omega|\vec{x}-\vec{y}|/c}\, \rho(\vec{y},\omega)}$$

$$\boxed{\vec{A}(\vec{x},\omega) = \frac{1}{c}\int d\vec{y}\, \frac{1}{|\vec{x}-\vec{y}|}\, e^{i\omega|\vec{x}-\vec{y}|/c}\, \vec{j}(\vec{y},\omega)}$$

where $\rho(\vec{y},\omega)$ and $\vec{j}(\vec{y},\omega)$ are the temporal Fourier transforms of the charge and current density.

4.2 Asymptotic potentials and fields

For large distances $r = |\vec{x}|$ from the sources, the expansion from §3 applies: $|\vec{x} - \vec{y}| \simeq r - \hat{x}\cdot\vec{y}$

$$\Rightarrow \varphi(\vec{x},\omega) = \int d\vec{y}\, \frac{1}{r}\left[1 + O(1/r)\right] e^{i\omega\frac{1}{c}(r - \hat{x}\cdot\vec{y} \ldots)} \rho(\vec{y},\omega)$$

$$= \frac{1}{r} e^{i\omega r/c} \int d\vec{y}\, e^{-i\omega\,\hat{x}\cdot\vec{y}/c \ldots} \rho(\vec{y},\omega) + O(1/r^2)$$

def.: $\boxed{\vec{k} = \frac{\omega}{c}\hat{x}}$ is called wave vector

remark: (1) This is consistent with ch 3 §1.5 remark (1).

(2) Far from the source the wave fronts are approximately plane $\Rightarrow$ ch 3 applies.

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$$\Rightarrow \varphi(\vec{x},\omega) = \frac{1}{r} e^{ikr} \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}} \rho(\vec{y},\omega)$$

$$= \frac{1}{r} e^{ikr} \rho(\vec{k},\omega)$$

with $\rho(\vec{k},\omega)$ the spatial Fourier transform of $\rho(\vec{y},\omega)$ and $k = |\vec{k}|$

Analogously,

$$\vec{A}(\vec{x},\omega) = \frac{1}{r} e^{ikr} \frac{1}{c}\vec{j}(\vec{k},\omega)$$

proposition: For far from the sources, the fields are given by

$$\boxed{\vec{B}(\vec{x},\omega) \simeq i\frac{\omega}{c}\,\frac{e^{i\omega r/c}}{r}\,\hat{x}\times\frac{1}{c}\vec{j}(\vec{k},\omega)}$$

$$\boxed{\vec{E}(\vec{x},\omega) \simeq -\hat{x}\times\vec{B}(\vec{x},\omega)}$$

remark: (3) The expression for $\vec{E}$ in terms of $\vec{B}$ follows instantly from the proposition in §3.1

proof: $\vec{B}(\vec{x},\omega) = \vec{\nabla}\times\vec{A}(\vec{x},\omega)$

$$\to \underline{B_\ell(\vec{x},\omega)} = \varepsilon_{\ell mn}\,\partial_m A_n(\vec{x},\omega)$$

$$= \varepsilon_{\ell mn}\left(\partial_m \frac{1}{r}e^{ikr}\right)\frac{1}{c}\, j_n(\vec{k},\omega)$$

$$\partial_m \frac{1}{r} = -\frac{1}{2}\frac{1}{r^3}\,2x_m = -\frac{x_m}{r^3} = -\frac{\hat{x}_m}{r^2} = O(1/r^2)$$

$$\frac{1}{r}\partial_m e^{ikr} = \frac{e^{ikr}}{r}\, ik\,\partial_m r = \frac{e^{ikr}}{r}\, ik\,\frac{1}{2}\frac{1}{r}\,2x_m$$

$$= ik\frac{e^{ikr}}{r}\,\hat{x}_m = O(1/r)$$

$$= \varepsilon_{\ell mn}\, ik\,\frac{e^{ikr}}{r}\,\hat{x}_m\,\frac{1}{c}\, j_n(\vec{k},\omega)$$

$$= ik\,\frac{e^{ikr}}{r}\left(\hat{x}\times\frac{1}{c}\vec{j}(\vec{k},\omega)\right)_\ell$$

and $\vec{E} = -\hat{x}\times\vec{B}$ follows from §3.1 prop., see remark (3)

4.2 The spectral distribution of the radiated energy

Theorem: The total energy radiated by the source per solid angle $d\Omega$ and frequency interval $d\omega$ is

$$\boxed{\frac{d^2U}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3}\left|\hat{x}\times\vec{j}(\vec{k},\omega)\right|^2}$$

remark: (1) Check a static source: $\vec{j}(\vec{k},t) = \vec{j}(\vec{k},\cancel{t}) \equiv \vec{j}(\vec{k})$

$$\to \vec{j}(\vec{k},\omega)\propto\delta(\omega) \to d^2U/d\Omega\, d\omega = 0 \ \checkmark$$

proof: The instantaneous flux of energy is given by the Poynting vector cf. §3.6:

$$\vec{P}(\vec{x},t) = \frac{c}{4\pi}\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)$$

$\to$ The total energy U radiated into a solid angle $d\Omega$ is (see §3.2)

$$\underline{\frac{dU}{d\Omega}} = \int dt\; r^2\,\hat{x}\cdot\vec{P}(\vec{x},t)$$

$$= \int dt\, r^2\, \hat{x} \cdot \frac{c}{4\pi}\left(\vec{E}(\vec{x},t) \times \vec{B}(\vec{x},t)\right)$$

$$= \frac{c}{4\pi} r^2 \int dt\, \hat{x} \cdot \left(\int \frac{d\omega}{2\pi} e^{-i\omega t} \vec{E}(\vec{x},\omega) \times \int \frac{d\omega'}{2\pi} e^{-i\omega' t} \vec{B}(\vec{x},\omega') \right)$$

$$= \frac{c}{4\pi} r^2 \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi}\, \hat{x} \cdot \left(\vec{E}(\vec{x},\omega) \times \vec{B}(\vec{x},\omega') \right) \underbrace{\int dt\, e^{-i(\omega+\omega')t}}_{= 2\pi\, \delta(\omega+\omega')}$$

$$= \underline{\frac{c}{4\pi} r^2 \int \frac{d\omega}{2\pi}\, \hat{x} \cdot \left(\vec{E}(\vec{x},\omega) \times \vec{B}(\vec{x},-\omega) \right)}$$

But $B_i(\vec{x},t) \in \mathbb{R} \rightarrow B_i(\vec{x},-\omega) = \int dt\, e^{-i\omega t} B_i(\vec{x},t) = \left(\int dt\, e^{i\omega t} B_i(\vec{x},t)\right)^*$

$$= B_i(\vec{x},\omega)^*$$

§4.2 prop

$$\rightarrow \underline{\frac{dU}{d\Omega}} = -\frac{c}{4\pi} r^2 \int \frac{d\omega}{2\pi}\, \hat{x} \cdot \left((\hat{x} \times \vec{B}(\vec{x},\omega)) \times \vec{B}(\vec{x},\omega)^* \right) \overset{\hat{x} \perp \vec{B}}{=} \frac{c}{8\pi^2} \int d\omega\, |\vec{B}(\vec{x},\omega)|$$

$$= \frac{c r^2}{4\pi^2} \int_0^\infty d\omega\, |\vec{B}(\vec{x},\omega)|^2$$

§4.2 prop

$$= \frac{c r^2}{4\pi^2} \int_0^\infty d\omega\, \frac{\omega^2}{c^2} \frac{1}{r^2} \frac{1}{c^2} \left| \hat{x} \times \vec{j}(\vec{k},\omega) \right|^2 = \underline{\frac{1}{4\pi^2 c^3} \int_0^\infty d\omega\, \omega^2 \left| \hat{x} \times \vec{j}(\vec{k},\omega) \right|}$$

$$\rightarrow \underline{\underline{\frac{d^2 U}{d\Omega\, d\omega}}} = \underline{\underline{\frac{\omega^2}{4\pi^2 c^3} \left| \hat{x} \times \vec{j}(\vec{k},\omega) \right|^2}} \qquad \square$$

4.4 Spectral distribution for dipole radiation

§4.3 $\rightarrow$ $d^2U/d\omega d\Omega$ is given by the Fourier transform of the current density

$$\vec{j}(\vec{k},\omega) \qquad \text{where} \qquad k \equiv |\vec{k}| = \omega/c = 2\pi/\lambda$$

with λ the wavelength of the radiation.

Consider small sources in the sense that $|\vec{y}| \ll \lambda$.

example: (1) For an atom radiating visible light, we have

$|\vec{y}| \leq$ a few Å

$\lambda \simeq$ thousands of Å

$$\to \quad \vec{j}(\vec{k},\omega) = \int d\vec{y}\, e^{-i\vec{k}\vec{y}} \int dt\, e^{i\omega t}\, \vec{j}(\vec{y},t)$$

$$= \int d\vec{y}\, [1 - i\vec{k}\cdot\vec{y} + \dots] \int dt\, e^{i\omega t}\, \vec{j}(\vec{y},t)$$

$$= \int dt\, e^{i\omega t} \int d\vec{y}\, \vec{j}(\vec{y},t) + O(a/\lambda)$$ with a the linear dimension of the source

§34 here

$$= \int dt\, e^{i\omega t} \frac{d}{dt}\vec{d}(t) + O(a/\lambda)$$

$$= -i\omega\, \vec{d}(\omega) + O(a/\lambda)$$

proposition: If a is the linear dimension of the source, and λ is the wavelength of the radiation, then to lowest order in $a/\lambda \ll 1$ the energy radiated per unit solid angle and unit frequency is

$$\boxed{\frac{d^2U}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} \sin^2\vartheta\, |\dot{\vec{d}}(\omega)|^2}$$

where ϑ is the angle between $\vec{d}$ and $\hat{x}$, and $\dot{\vec{d}}(\omega)$ is the temporal Fourier transform of $\dot{\vec{d}}(t)$.

proof: In dipole approximation, $\vec{j} \parallel \vec{d} \to |\hat{x}\times\vec{j}|^2 = \sin^2\vartheta\, |$ □

corollary: The total radiated energy per frequency is

$$\boxed{\frac{dU}{d\omega} = \frac{2}{3}\frac{\omega^2}{\pi c^3} |\dot{\vec{d}}(\omega)|^2}$$

proof: $\int d\Omega \sin^2\vartheta = 2\pi \int_{-1}^{1} d\eta\, (1-\eta^2) = 2\pi\, 2\cdot\frac{2}{3} = \frac{8\pi}{3}$ □

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[illegible] 12 (§ 37, 38, 39)

example (2) A point charge e with trajectory $\vec{y}(t)$ and velocity $\vec{v}(t) = \dot{\vec{y}}(t) \ll c$

$$\to \vec{j}(\vec{y},t) = e\,\vec{v}(t)\,\delta(\vec{y}-\vec{y}(t))$$

→ Either the x-axis or the y-axis must be in the equatorial plane → The $\cos\vartheta$ must be in the 1st or 2nd entry, but NOT in the 3rd!

Now modulate with $\sin^2\vartheta$:

→ $(\cos\vartheta \sin^2\vartheta, \sin\vartheta\sin\varphi \sin^2\vartheta, \sin\vartheta\cos\varphi \sin^2\vartheta)$ $(0\le\vartheta\le\pi, 0\le\varphi<2\pi)$

parametrizes the surface of constant intensity, or, equivalently,

$(\sin\vartheta\sin\varphi\sin^2\vartheta, \cos\vartheta\sin^2\vartheta, \sin\vartheta\cos\varphi\sin^2\vartheta)$

$\{4.?\}$ → $\frac{d^2I}{d\omega d\Omega} \propto \omega^2 |\hat{x}\times\vec{j}(\omega)|^2 = |\hat{x}\times\dot{\vec{j}}(\omega)|^2$

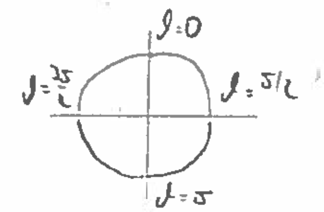
and $\vec{j}(\omega) \propto \omega\vec{d}(\omega) \propto \dot{\vec{d}}(\omega)$ → $\dot{\vec{j}}(\omega) \propto \ddot{\vec{d}}(\omega)$

and $\ddot{\vec{d}}(\omega) \propto \dot{\vec{v}}(\omega)$

→ No radiation in the direction of $\dot{\vec{v}}$

If the radiation were isotropic, the polar diagram would be a circle, which is parametrized by

$$(x,y) = (\sin\vartheta, \cos\vartheta) \quad (0\le\vartheta<2\pi)$$



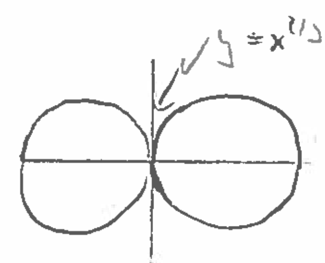
Now this gets modulated by $\sin^2\vartheta$

→ The contour of equal intensity in the v-$\dot{v}$ plane is

$$(x,y) = (\sin\vartheta\cdot\sin^2\vartheta, \cos\vartheta\cdot\sin^2\vartheta)$$

with $\vartheta=0$ corresponding to the $\dot{\vec{v}}$-direction.

For $\vartheta\to 0$, $x\approx\vartheta^3$, $y\approx\vartheta^2$ → $y(x\to 0) = x^{2/3}$



Now we rotate out of the orbital plane. For isotropic radiation the surface of constant intensity would be a sphere:

$$(\cos\vartheta, \sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi) \quad (0\le\vartheta\le\pi, 0\le\varphi\le 2\pi)$$

We can rotate the sphere out of the plane

→

Problem 43
Radiation by an accelerated point particle
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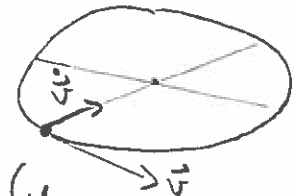
§3.4 here $\to \dot{\vec d}(t) = \int d^3y\, \vec j(\vec y, t) = e\vec v(t) \to \dot{\vec d}(\omega) = e\vec v(\omega)$

$$\to \underline{\underline{\frac{dU}{d\omega}}} = \frac{2}{3}\frac{e^2\omega^2}{\pi c^3}|\vec v(\omega)|^2 = \underline{\underline{\frac{2}{3}\frac{e^2}{\pi c^3}|\dot{\vec v}(\omega)|^2}}$$

remark: (1) $dU/d\omega$ is given by the temporal Fourier transform of the acceleration

(2) This is consistent with the Larmor formula, §3.3, see Problem 43.

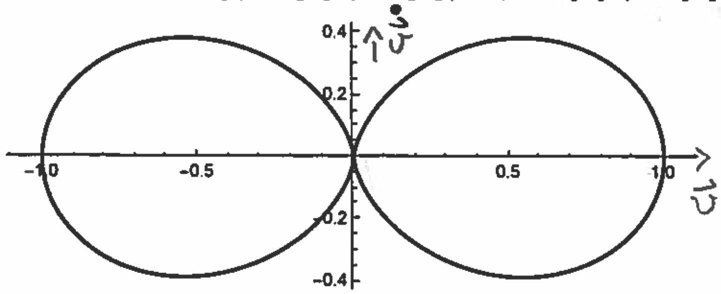
example: (3) Suppose the charge moves slowly ($v \ll c$) on a circle $\to \dot{\vec v}$ is purely radial



$\to$ * Power is maximal in the direction $\perp$ to $\dot{\vec v}$ ($\vartheta = \pm\pi/2$)

```
In[11]:= ParametricPlot[{Sin[x] (Sin[x])^2, Cos[x] (Sin[x])^2}, {x, 0, 2 Pi}]
```

Out[11]=



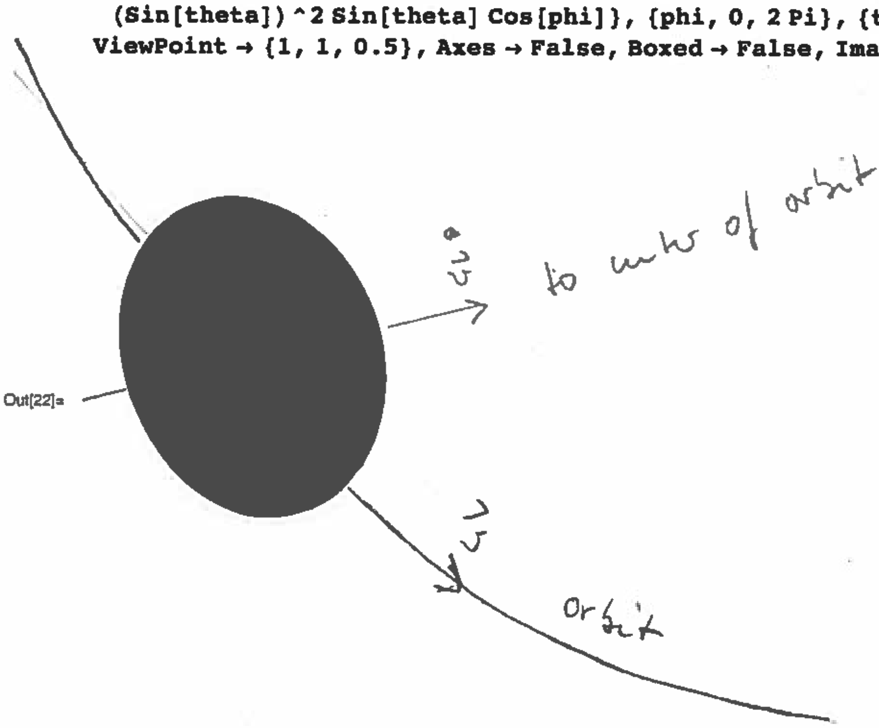
* No radiation is emitted in the direction of $\dot{\vec v}$ ($\vartheta = 0$) $\forall$ see f.p.

* In the orbital plane the radiation intensity has a 2-lobed structure $\not\equiv$ see f.p.

* In 3-d it has the shape of a torus: $\not\equiv$ see f.p.

```
In[22]:= ParametricPlot3D[{(Sin[theta])^2 Cos[theta], (Sin[theta])^2 Sin[theta] Sin[phi],
    (Sin[theta])^2 Sin[theta] Cos[phi]}, {phi, 0, 2 Pi}, {theta, 0, Pi},
   ViewPoint → {1, 1, 0.5}, Axes → False, Boxed → False, ImageSize → Scaled[0.3]]
```

Out[22]=



4.5 Example: Radiation by a damped harmonic oscillator

Consider a charged particle in a harmonic potential (oscillation frequency ω_0) with damping constant γ. Eq. of motion: $\boxed{\ddot{y} = -\omega_0^2 y - \gamma \dot{y}}$ (*)

remark: (1) We think of the damping as due to the radiation emitted by the oscillator.

(2) This is a simple model for a classical electron in an atom.

initial conditions: $y(t=0) = a$, $\dot{y}(t=0) = 0$

know: For weak damping, $\gamma \ll \omega_0$, the solution of (*) is

$$\boxed{y(t) \simeq a \cos\omega_0 t \, e^{-\gamma t/2}} \quad (t > 0)$$

proof: Mechanics course or Problem 44

→ The velocity is $v(t) = -a\omega_0 \sin\omega_0 t \, e^{-\gamma t/2}\left[1 + O(\gamma/\omega_0)\right]$

with Fourier trafo $v(\omega) \simeq -a\omega_0 \int_0^\infty dt \, e^{i\omega t} \sin\omega_0 t \, e^{-\gamma t/2}$

$$= \frac{-a\omega_0}{2i}\int_0^\infty dt \left[e^{i\omega t + i\omega_0 t - \gamma t/2} - e^{i\omega t - i\omega_0 t - \gamma t/2}\right]$$

$$= \frac{-a\omega_0}{2i}\left[\frac{-1}{i(\omega+\omega_0) - \gamma/2} - \frac{-1}{i(\omega-\omega_0) - \gamma/2}\right]$$

$$= \frac{\cancel{-}a\omega_0}{2\cancel{i}}\left[-\frac{\cancel{i}}{\omega+\omega_0+i\gamma/2} + \frac{\cancel{i}}{\omega-\omega_0+i\gamma/2}\right]$$

$$= \frac{a\omega_0}{2}\left[\frac{1}{\omega-\omega_0+i\gamma/2} - \frac{1}{\omega+\omega_0+i\gamma/2}\right]$$

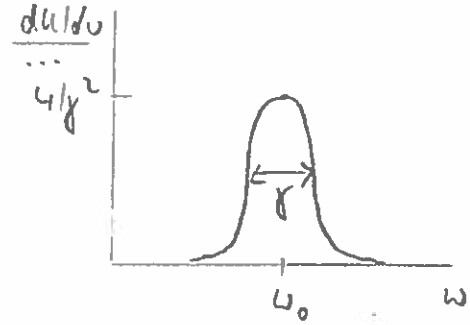
Let $\omega > 0$ (discussion for $\omega < 0$ is analogous)

→ $v(\omega)$ is dominated by the first term for $\omega \simeq \omega_0$

→ $|v(\omega)|^2 \simeq \frac{a^2\omega_0^2}{4} \frac{1}{(\omega-\omega_0)^2 + \gamma^2/4}$

→ $dW/d\omega = \frac{2e^2}{3\pi c^3}|\dot{v}(\omega)|^2 \simeq \frac{2e^2}{3\pi c^3}\frac{a^2\omega_0^2}{4}\frac{\omega^2}{(\omega-\omega_0)^2+\gamma^2/4}$

$$= \underline{\underline{\frac{e^2 a^2 \omega_0^4}{6\pi c^3} \frac{1}{(\omega-\omega_0)^2+\gamma^2/4}}} \qquad \text{for } \underline{\omega \simeq \omega_0}$$



discussion. (1) Spectrum is a Lorentzian centered about $\omega = \omega_0$ with width γ.

(2) Consider the total energy radiated:

$\omega<0 + \omega>0$

$$\underline{U} = 2\int_0^\infty d\omega \frac{dU}{d\omega} = 2\frac{e^2a^2\omega_0^4}{6\pi c^3}\int_0^\infty d\omega \frac{1}{(\omega-\omega_0)^2+\gamma^2/4} = 2\frac{e^2a^2\omega_0^4}{6\pi c^3}\int_{-\omega_0}^\infty d\omega \frac{1}{\omega^2+\gamma^2/4}$$

$$= 2\frac{e^2a^2\omega_0^4}{6\pi c^3}\frac{2}{\gamma}\int_{-2\omega_0/\gamma}^\infty dx \frac{1}{x^2+1} \simeq 2\frac{e^2a^2\omega_0^4}{6\pi c^3}\frac{2}{\gamma}\underbrace{\int_{-\infty}^\infty dx \frac{1}{1+x^2}}_{=\pi} = \underline{\frac{2e^2a^2\omega_0^4}{3c^3\gamma}}$$

and compare with the initial energy of the oscillator:

$$U_{osc}^{t=0} = \frac{m}{2}\omega_0^2 a^2$$

$$\rightarrow \quad \underline{U} = U_{osc}^{t=0}\,\frac{2}{m}\frac{1}{\cancel{\omega_0^2 a^2}}\frac{2e^2\cancel{a^2}\omega_0^4}{3c^3\gamma} = \underline{U_{osc}^{t=0}\,\frac{4}{3}\frac{e^2\omega_0^2}{mc^3\gamma}}$$

But the oscillator energy $U_{osc}^{t=0}$ must have gone into the radiated energy U $\rightarrow$

Problem 44
classical model of an atom

$$\underline{\underline{\gamma = \frac{4}{3}\frac{e^2\omega_0^2}{mc^3}}}$$

(3) Compare with Problem 39, which calculated the radiated power and concluded $U_{osc} = U_{osc}^{t=0} e^{-t/\tau}$ where $\frac{1}{\tau} = \frac{1}{2}\gamma$ with γ as calculated above. $\rightarrow$ The two approaches are consistent.

(4) See Problem 44 for a more thorough discussion of the approximations made above.

§5 Čerenkov radiation

5.1 The time-Wigner function, and the macroscopic power spectrum

§4.3 → The spectral distribution for radiation from a time-dependent current density is

$$\frac{d^2I}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \left| \hat{x} \times \vec{j}(\vec{k},\omega) \right|^2$$

$$= \frac{\omega^2}{4\pi^2 c^3} \left(\hat{x} \times \int dt\, e^{i\omega t}\, \vec{j}(\vec{k},t) \right) \cdot \left(\hat{x} \times \int dt'\, e^{-i\omega t'}\, \vec{j}(\vec{k},t')^* \right)$$

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$$= \frac{\omega^2}{4\pi^2 c^3}\, \varepsilon_{ijk}\, \varepsilon_{ilm}\, \hat{x}_j \hat{x}_l \int dt\, dt'\, e^{i\omega(t-t')}\, j_k(\vec{k},t)\, j_m(\vec{k},t')^*$$

Consider

$$\int dt\, dt'\, j_k(\vec{k},t)\, j_m(\vec{k},t')^*\, e^{i\omega(t-t')} = \left[\begin{array}{l} t = T + \tau/2 \\ t' = T - \tau/2 \end{array} \right]$$

$$= \int dT \int d\tau\, e^{i\omega\tau}\, j_k(\vec{k}, T+\tau/2)\, j_m(\vec{k}, T-\tau/2)^*$$

$$= \int dT \int d\tau\, e^{i\omega\tau}\, W_{km}(\vec{k}; T, \tau)$$

where $W_{km}(\vec{k}; T, \tau) := j_k(\vec{k}, T+\tau/2)\, j_m(\vec{k}, T-\tau/2)^*$

remark: (1) W_{kk} is an example of what is called a (time) Wigner function. It separates two times into an "average", or "macroscopic" time T and a "relative" or "microscopic" time τ.

(2) Only relative times $|\tau| \lesssim 1/\omega$ will appreciably contribute to the τ-integral, whereas all times T during which the source is active will contribute to the T-integral.

(3) This makes sense if the two time scales are well separated. E.g., a laser pulse of duration $T \gg 1/\omega$

<u>def.:</u> The spectral distribution at time T,

$$\boxed{\frac{d^2 P(T)}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3}\, \varepsilon_{ijk}\varepsilon_{ilm}\, \hat{x}_j \hat{x}_l \int d\tau\, e^{i\omega\tau}\, W_{km}(\vec{k};T,\tau)}$$

is called the <u>macroscopic power spectrum</u>

<u>remark:</u> (4) The spectral distribution of the radiated energy is given by

$$\frac{d^2 U}{d\omega\, d\Omega} = \int dT\, \frac{d^2 P(T)}{d\omega\, d\Omega}$$

5.2 Čerenkov radiation

Consider a point particle as in § 3.3:

$$\vec{j}(\vec{y},t) = e\,\vec{v}(t)\,\delta(\vec{y}-\vec{R}(t))$$

and specialize to uniform motion along a straight trajectory:

$$\vec{R}(t) = \vec{v}\,t \quad , \quad \vec{v}(t) \equiv \vec{v} = \text{const.}$$

<u>remark:</u> (1) We know that in vacuum this will not result in radiation.

$$\rightarrow \underline{\vec{j}(\vec{k},t)} = \int d\vec{y}\, e^{-i\vec{k}\cdot\vec{y}}\, e\vec{v}\,\delta(\vec{y}-\vec{v}t) = e\vec{v}\, e^{-i\vec{k}\cdot\vec{v}t}$$

$$\underset{\vec{k}=\hat{x}\omega/c}{=} \underline{e\vec{v}\, e^{-i\hat{x}\cdot\vec{v}t\omega/c}}$$

$$\rightarrow \underline{W_{km}(\vec{k};T,\tau)} = e^2 v_k v_m\, e^{-i\hat{x}\cdot\vec{v}(T+\tau/2)\omega/c + i\hat{x}\cdot\vec{v}(T-\tau/2)\omega/c}$$

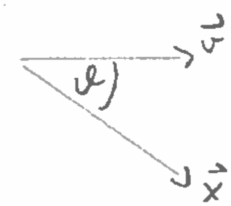
$$= \underline{e^2 v_k v_m\, e^{-i\hat{x}\cdot\vec{v}\,\omega\tau/c}}$$

remark: (2) The Wigner fct. is independent of T, as one would expect for uniform motion.

$$\rightarrow \quad \frac{d^2 P(T)}{d\omega\, d\Omega} = \frac{\omega^2 e^2}{4\pi^2 c^3} \underbrace{\varepsilon_{ijk}\varepsilon_{ilm}\, \hat{x}_j \hat{x}_l v_k v_m}\int d\tau\, e^{i\omega\tau}\, e^{-i\hat{x}\cdot\vec{v}(\omega/c)\tau}$$

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})\hat{x}_j \hat{x}_l v_k v_m = \vec{v}^2 - (\hat{x}\cdot\vec{v})^2$$

$$= v^2 \sin^2\vartheta$$



$$= \frac{\omega^2 e^2}{4\pi^2 c}\left(\frac{v}{c}\right)^2 \sin^2\vartheta \int d\tau\, e^{i\omega(1-\frac{v}{c}\cos\vartheta)\tau}$$

$$= \frac{\omega^2 e^2}{4\pi^2 c}\left(\frac{v}{c}\right)^2 \sin^2\vartheta\ \delta\left(\omega\left(1-\frac{v}{c}\cos\vartheta\right)\right)$$

$$= \frac{|\omega| e^2}{4\pi^2 c}\left(\frac{v}{c}\right)^2 \sin^2\vartheta\ \delta\left(1-\frac{v}{c}\cos\vartheta\right)$$

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remark: (3) $v/c < 1$, $\cos\vartheta < 1$ $\rightarrow$ $1-\frac{v}{c}\cos\vartheta > 0$

$\rightarrow$ no radiation, in agreement with § 3.3

unless $v/c > 1$ ("tachyonic particles")

(4) In matter, $c \rightarrow c/n$ with n the index of refraction

$\rightarrow$ $v/c \rightarrow vn/c$ and $vn/c > 1$ is possible!

(5) Strictly speaking, this requires a theory of electromagnetic waves in continuous matter. Here we assume that $c \rightarrow c/n$ † suffices to catch the main effects. Also keep in mind: We are applying a nonrelativistic approximation to a situation where v/c is no longer small (see Problem 45).

(6) n is frequency dependent, so we should write $n(\omega)$

† plus $e^2 \rightarrow e^2/n^2$ *

* This is because in a dielectric the charge gets screened, and the Coulomb force law is $F = \frac{e^2}{\varepsilon r^2}\hat{r}$ with ε the dielectric constant, and $n = \sqrt{\varepsilon}$.

The total radiated Power is

$$\mathcal{P} = \frac{dE}{dt} = \int d\omega \, \frac{d\mathcal{P}}{d\omega} = \frac{\cancel{2}e^2 v}{\cancel{2}\pi c^2} \int_0^\infty d\omega \, \omega \left(1 - \frac{c^2}{n^2(\omega) v^2}\right) \Theta\left(\frac{c^2}{n^2(\omega) v^2} < 1\right)$$

$v = dx/dt$

$$\leadsto \frac{dE}{dx} = \frac{e^2}{\pi c^2} \int_0^\infty d\omega \, \omega \left(1 - \frac{c^2}{n^2(\omega) v^2}\right) \Theta\left(\frac{c^2}{n^2(\omega) v^2} < 1\right)$$

remark: (a) The human eye counts photons, and each photon has an energy $\hbar\omega$ $\leadsto$ The number of photons per frequency and distance is

$$\frac{dN}{dx\,d\omega} = \frac{\alpha}{\pi c}\left(1 - \frac{c^2}{n^2(\omega) v^2}\right)$$ with $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$

the fine structure constant

A typical reactor electron travels at $v \approx 0.9c$ and $n(\omega)$ for water decreases monotonically throughout the visible range, see p91c $\leadsto$ The Cerenkov radiation observed in a water-cooled reactor favors blue.

$$\rightarrow \quad \boxed{\frac{d^2 P(i)}{d\omega\, d\Omega} = \frac{|\omega| e^2/n^2}{4\pi^2 c/n}\left(\frac{vn}{c}\right)^2 \sin^2\theta\, \delta\!\left(1 - \frac{vn}{c}\cos\theta\right)}$$

<u>interpretation</u>: A particle moving through a medium faster than the speed of light in that medium emits radiation ("Čerenkov radiation", although first observed by Mallet), on a cone with angle θ where $\cos\theta = \frac{c}{vn}$

<u>proposition</u>: The total power spectrum of the Čerenkov radiation is

$$\boxed{\frac{dP}{d\omega} = |\omega| \frac{e^2 v}{2\pi c^2}\left(1 - \frac{c^2}{n^2(\omega) v^2}\right)}$$

where $n(\omega)$ is the index of refraction for light with frequency ω.

<u>proof</u>:

$$\frac{dP}{d\omega} = \int d\Omega\, \frac{d^2 P}{d\omega\, d\Omega} = \frac{|\omega| e^2}{4\pi^2 c n}\left(\frac{vn}{c}\right)^2 2\pi \int_{-1}^{1} d\eta\, (1-\eta^2)\, \delta\!\left(1 - \frac{vn}{c}\eta\right)$$

$$= \frac{|\omega| e^2}{2\pi c n}\left(\frac{vn}{c}\right)^2 \frac{c}{vn} \int_{-1}^{1} d\eta\, (1-\eta^2)\, \delta\!\left(\eta - \frac{c}{vn}\right)$$

$$= |\omega| \frac{e^2 v}{2\pi c^2}\left(1 - \frac{c^2}{n^2(\omega) v^2}\right) \quad \square$$

Problem 45

Cherenkov radiation

Week 4

Problset 13 (Limiting Needle exp.)

<u>remark</u>: (7) This is nonzero only for the (finite) frequency range where $v n(\omega) > c$ $\rightarrow$ The total radiated power $P = \int d\omega\, dP/d\omega$ is finite!

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(8) This is the radiated energy per time and frequency. A Čerenkov counter observes the energy radiated per distance traveled by the particle $\rightarrow$ calculate that

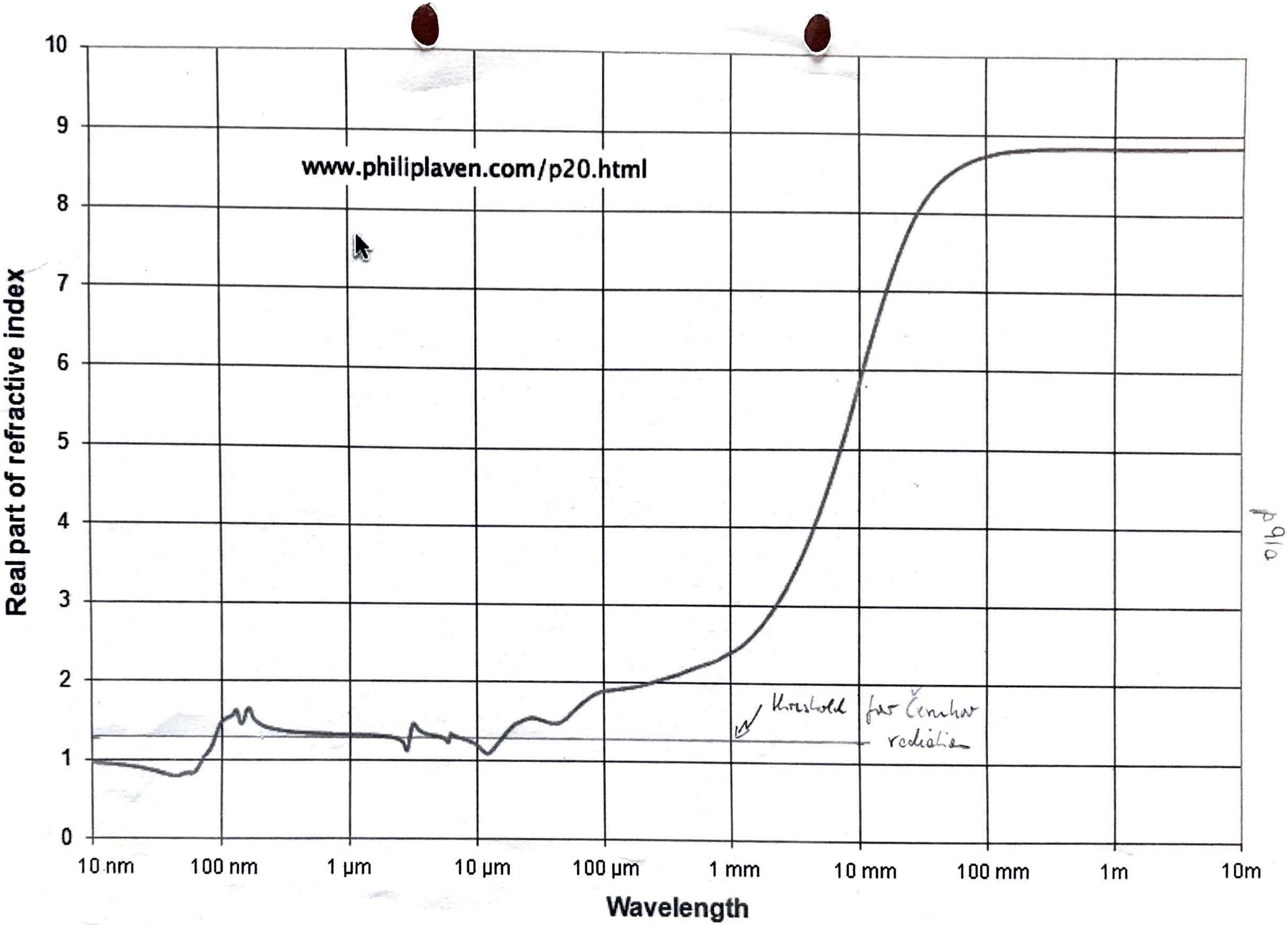
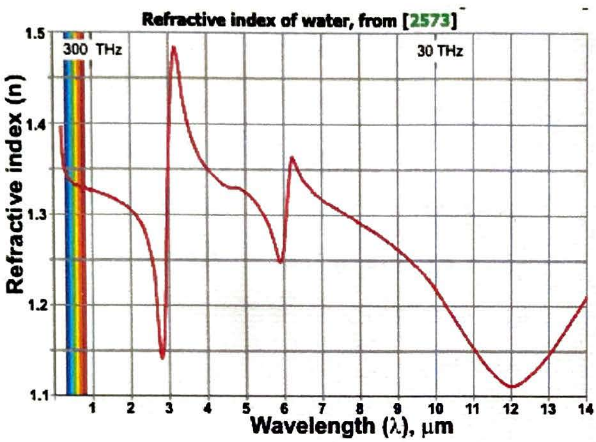


Fig. 7 Segelstein's values for the real part of the refractive index of water for wavelengths from 10 nm to 10m

However, this is a rather small effect:

Here is the index or refraction of water as a function of the wavelength (from http://www1.lsbu.ac.uk/water/dielectric_constant.html#refract)



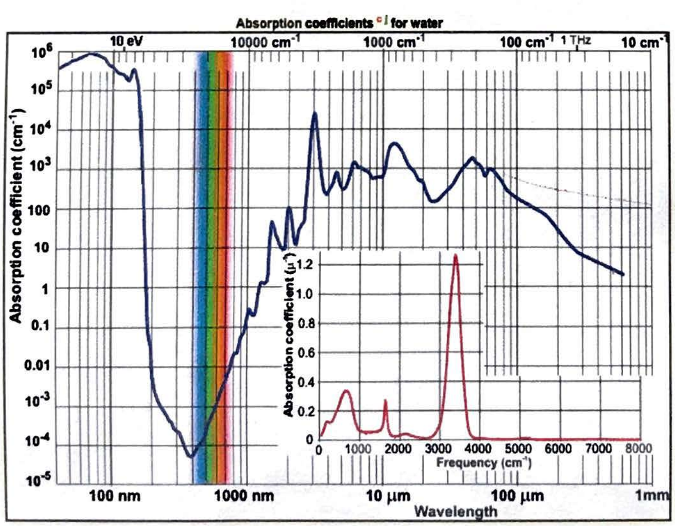
n in the blue is about 1.35, and n in the red is about 1.33, and (1.35/1.33)^2 = 1.03. So the frequency dependence of the index of refraction favors the blue end of the visible spectrum, but this is a rather small effect:

$$\frac{dN/dx\,d\omega\,|_{blue}}{dN/dx\,d\omega\,|_{red}} = \frac{1-\frac{c^2}{v^2n^2(\omega)}\big|_{blue}}{1-\frac{c^2}{v^2n^2(\omega)}\big|_{red}} \simeq 1.07$$

A larger effect is the frequency dependence of the absorption coefficient:

And here is the absorption coefficient (from http://www1.lsbu.ac.uk/water/water_vibrational_spectrum.html)

The visible and UV spectra of liquid water



Assume that the Čerenkov photons run through 1m of water before emerging into air. Then virtually all of the blue photons will make it, but only about 1/e = 0.37 of the red ones do!

Conclusion: The chief reason for the blue color of the Čerenkov radiation is the frequency dependence of the absorption coefficient of water. The frequency dependence of the index of refraction also favors the blue end of the spectrum, but that's a much smaller effect.

§6 Synchrotron radiation

idea: Discuss motion of a charged particle in a homogeneous B-field, as in Problem 38, but

* consider relativistic motion, and
* discuss the spectrum

6.1 Relativistic motion of a charged particle in a homogeneous B-field

Consider a charged particle (charge e, rest mass m) in a homogeneous B-field.

PHYS 611 $\rightsquigarrow$ $$\boxed{\frac{d\vec{p}}{dt} = \frac{e}{c}\,\vec{v}\times\vec{B}} \quad (*)$$

with $\vec{p} = \gamma m \vec{v}$ the momentum $\left(\gamma = 1/\sqrt{1-v^2/c^2}\right)$

remark: (1) $(*)$ is true for both relativistic and nonrelativistic motion

(2) Force is purely transverse $\rightsquigarrow$ $E = \gamma m c^2 = \text{const}$

and $\vec{p} = \frac{E}{c^2}\vec{v}$ with E the particle's energy.

$\rightsquigarrow$ The eq. of motion can be written

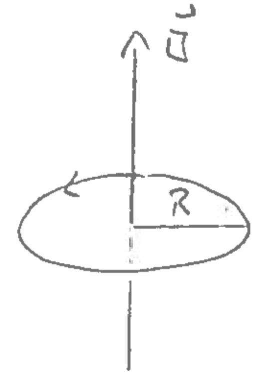
$$\frac{E}{c^2}\frac{d\vec{v}}{dt} = \frac{e}{c}\,\vec{v}\times\vec{B} \quad \rightsquigarrow \quad \frac{d\vec{v}}{dt} = \frac{ec}{E}\,\vec{v}\times\vec{B} = -\frac{ecB}{E}\,\hat{B}\times\vec{v}$$

def.: $\omega_0 = \frac{|e|cB}{E}$ is called Larmor frequency

remark: (3) For nonrelativistic particles, $\omega_0 \approx \frac{|e|cB}{mc^2} = \frac{|e|B}{mc}$ is called cyclotron frequency.

5/1/17

initial condition: $\vec{v}\perp\vec{B}$ $\rightsquigarrow$ $\vec{v}\perp\vec{B}$ for all times.



conclusion: The particle moves on a circle of radius

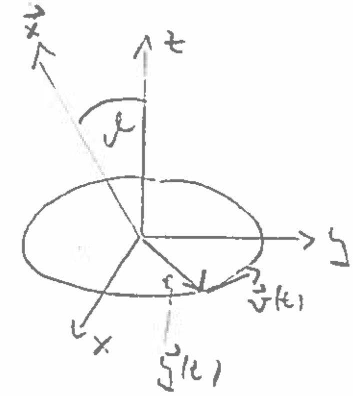
$$\boxed{R = \frac{v}{\omega_0} = \frac{v}{c}\,\frac{E}{|e|B}}$$

and the momentum is related to the radius by

$$\boxed{p = \frac{E}{c^2}\,v = \frac{1}{c}\,|e|\,B\,R}$$

remark: (4) This provides an easy method for measuring the momentum of a relativistic particle.

6.2 The power spectrum of synchrotron radiation



Consider motion in the x-y-plane with an observer at point $\vec x$ and $\vartheta = \sphericalangle(\vec x, \hat z)$

Choose coordinate system such that $\vec x = (x, 0, z)$

$\rightarrow$ $\boxed{\hat x = (\sin\vartheta, 0, \cos\vartheta)}$

and initial conditions such that $\vec y(t) = R(\cos\omega_0 t, \sin\omega_0 t, 0)$

$\rightarrow$ $\boxed{\vec v(t) = v(-\sin\omega_0 t, \cos\omega_0 t, 0)}$ where $v = R\omega_0$

$\rightarrow$ The current density is

$$\vec j(\vec y, t) = e\,\vec v(t)\,\delta(\vec y - \vec y(t))$$

$$\rightarrow \vec j(\vec k, t) = \int d\vec y\, e^{-i\vec k\vec y}\, e\,\vec v(t)\,\delta(\vec y - \vec y(t)) = e\,\vec v(t)\, e^{-i\vec k\cdot\vec y(t)}$$

Lect 5 (1 lecture + written homework exp. discussion)

$$= e\,\vec v(t)\, e^{-i\frac{\omega}{c}\hat x\cdot\vec y(t)}$$

and the charge density,

Problem 14 (#40, 41, 42)

$$\rho(\vec k, t) = e\, e^{-i\frac{\omega}{c}\hat x\cdot\vec y(t)}$$

lemma 1: The power spectrum from §5.1 can be written

$$\boxed{\frac{d^2 P(T)}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau} \left[\vec{j}(\vec{k}, T+\tau/2)\cdot \vec{j}(\vec{k}, T-\tau/2)^* - c^2 \rho(\vec{k}, T+\tau/2)\, \rho(\vec{k}, T-\tau/2)^* \right]}$$

proof: §5.1 $\rightarrow$ the integrand (without the $e^{i\omega\tau}$) is

$$\underline{\varepsilon_{ijk}\, \varepsilon_{ilm}\, \hat{x}_j \hat{x}_l\, W_{km}(\vec{k}; T, \tau)} \overset{§5.2}{=} \vec{j}(\vec{k}, T+)\cdot \vec{j}(\vec{k}, T-)^* - \left(\hat{x}\cdot\vec{j}(\vec{k}, T+)\right)\left(\hat{x}\cdot\vec{j}(\vec{k}, T-)\right)^*$$

That $\hat{x} = \hat{k}$, and $k = \omega/c$

and the continuity eq. $\partial_t \rho(\vec{x}, t) = -\vec{\nabla}\cdot\vec{j}(\vec{x}, t)$

implies $i\omega\, \rho(\vec{k}, \omega) = i\vec{k}\cdot\vec{j}(\vec{k}, \omega) = i\frac{\omega}{c}\, \hat{x}\cdot\vec{j}(\vec{k}, \omega)$

$\rightarrow$ $\underline{\hat{x}\cdot\vec{j}(\vec{k}, t) = c\,\rho(\vec{k}, t)}$ $\square$

lemma 2: $\boxed{\vec{v}(T+\tau/2)\cdot\vec{v}(T-\tau/2) = v^2 \cos\omega_0\tau}$

proof: $\underline{\frac{1}{v^2}\vec{v}(T+)\cdot\vec{v}(T-)} = \sin(\omega_0 T+)\sin(\omega_0 T-) + \cos(\omega_0 T+)\cos(\omega_0 T-)$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$= -\frac{1}{2}\left(\cancel{\cos 2\omega_0 T} - \cos\omega_0\tau\right) + \frac{1}{2}\left(\cancel{\cos 2\omega_0 T} + \cos\omega_0\tau\right)$$
$$= \underline{\cos\omega_0\tau} \quad \square$$

lemma 3: $\boxed{e^{\mp i\frac{\omega}{c}\hat{x}\cdot\vec{r}(T\pm\tau/2)} = \sum_{n=-\infty}^{\infty} (\mp i)^n\, e^{\mp in\omega_0(T\pm\tau/2)}\, J_n\!\left(\frac{\omega}{c}R\sin\vartheta\right)}$

with $J_n(x)$ a Bessel function of the first kind.

proof: The Bessel fct. obey $e^{iz\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m e^{im\varphi} J_m(z)$

and $\hat{x}\cdot\vec{\xi}(t) = \sin\vartheta\, R\cos(\omega_0 t)$

$$\rightarrow \underline{e^{\mp i\frac{\omega}{c}\hat{x}\cdot\vec{\xi}(t)}} = e^{\mp i\underbrace{\frac{\omega}{c}R\sin\vartheta}_{=z}\cos\underbrace{\omega_0 t}_{=\varphi}} = \underline{\sum_{m=-\infty}^{\infty}(\mp i)^m e^{\mp im\omega_0 t} \times J_m\left(\frac{\omega}{c}R\sin\vartheta\right)}$$

$$\rightarrow \underline{\underline{\frac{d^2 P(T)}{d\omega\, d\Omega}}} = \frac{\omega^2 e^2}{4\pi^2 c^3}\int d\tau\, e^{i\omega\tau}\left[\vec{v}(T+\tau/2)\cdot\vec{v}(T-\tau/2) - c^2\right] e^{-i\frac{\omega}{c}\hat{x}\cdot\left[\vec{\xi}(T+\frac{\tau}{2}) - \vec{\xi}(T-\frac{\tau}{2})\right]}$$

$$= \frac{\omega^2 e^2}{4\pi^2 c}\int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right]$$

$$\times \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-i)^m e^{-im\omega_0(T+\tau/2)} J_m\left(\frac{\omega}{c}R\sin\vartheta\right)(i)^n e^{in\omega_0(T-\tau/2)} \times J_n\left(\frac{\omega}{c}R\sin\vartheta\right)$$

5/8/17

$$= \underline{\frac{\omega^2 e^2}{4\pi^2 c}\sum_{m,n=-\infty}^{\infty} e^{-i(m-n)\omega_0 T}(i)^{n-m} J_m\left(\frac{\omega}{c}R\sin\vartheta\right) J_n\left(\frac{\omega}{c}R\sin\vartheta\right) \times \int d\tau\, e^{i\omega\tau}\left[\frac{v^2}{c^2}\cos\omega_0\tau - 1\right] e^{-i(m+n)\omega_0\tau/2}}$$

remark: (1) For the macroscopic power spectrum we are not interested in how the emission varies on the microscopic time scale set by $1/\omega_0$ → Average over an oscillation period

Lemma 4. $\overline{e^{-i(m-n)\omega_0 T}} = \delta_{mn}$ where $\overline{f(T)}$ indicates a time average over one oscillation period.

proof: $\overline{e^{-i(m-n)\omega_0 T}} = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dT\, e^{-i(m-n)\omega_0 T} = \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-i(m-n)x}$

$$= \frac{1}{2\pi}\int_0^{2\pi} dx\left[\cos(m-n)x - i\sin(m-n)x\right] = \underline{\delta_{mn}}$$

$$\rightarrow \quad \overline{\frac{d^2 P(T)}{d\omega\, d\Omega}} = \frac{\omega^2 e^2}{4\pi^2 c}\sum_{m=-\infty}^{\infty}\left(J_m\left(\tfrac{\omega}{c}R\sin\vartheta\right)\right)^2 \int d\tau\, e^{i(\omega-m\omega_0)\tau} \times \left[\frac{v^2}{2c^2}\left(e^{i\omega_0\tau}+e^{-i\omega_0\tau}\right)-1\right]$$

$$= \frac{\omega^2 e^2}{4\pi^2 c}\sum_{m=-\infty}^{\infty}\left(J_m\left(\tfrac{\omega}{c}R\sin\vartheta\right)\right)^2\left[\frac{v^2}{2c^2}\Big(\delta(\omega-(m-1)\omega_0)+\delta(\omega-(m+1)\omega_0)\Big) - \delta(\omega-m\omega_0)\right]2\pi$$

$$= \frac{\omega^2 e^2}{2\pi c}\sum_{m=-\infty}^{\infty}\left[\frac{v^2}{2c^2}\left(J^2_{m+1}\left(\tfrac{\omega}{c}R\sin\vartheta\right)+J^2_{m-1}\left(\tfrac{\omega}{c}R\sin\vartheta\right)\right)-J_m^2\left(\tfrac{\omega}{c}R\sin\vartheta\right)\right]\times\delta(\omega-m\omega_0)$$

$\rightarrow 0$ via $\omega^2\delta(\omega)=0$

$$= \frac{\omega^2 e^2}{2\pi c}\left(\sum_{m=1}^{\infty}+\sum_{m=-1}^{-\infty}+\cancel{\sum\delta_{m0}}\right)[\dots]\,\delta(\omega-m\omega_0)$$

take $\omega>0$:

$$= \frac{\omega^2 e^2}{2\pi c}\sum_{m=1}^{\infty}\left[\frac{v^2}{2c^2}\left(J^2_{m+1}\left(\tfrac{\omega R}{c}\sin\vartheta\right)+J^2_{m-1}\left(\tfrac{\omega R}{c}\sin\vartheta\right)\right)-J_m^2\left(\tfrac{\omega R}{c}\sin\vartheta\right)\right]\times\delta(\omega-m\omega_0)$$

(1') The frequencies emitted are the Larmor(a) frequency and all of its harmonics!

(2) The weights of negative harmonics are the same as those of the positive ones since $J_{-m}(x)=(-)^m J_m(x)=J_m(-x)$ period, ...

Theorem: The macroscopic power spectrum averaged over a microscopic

$$\boxed{\frac{d^2 P}{d\omega\, d\Omega} = \sum_{m=1}^{\infty}\delta(\omega-m\omega_0)\,\frac{dP_m}{d\Omega}}$$

with the power radiated into the m^{th} harmonic given by

$$\boxed{\frac{dP_m}{d\Omega} = \frac{\omega_0 e^2}{\pi R}\left(\frac{v}{c}\right)^3 m^2\left\{\left[J_m'\left(m\tfrac{v}{c}\sin\vartheta\right)\right]^2+\left[\frac{J_m\left(m\tfrac{v}{c}\sin\vartheta\right)}{\tfrac{v}{c}\operatorname{tg}\vartheta}\right]^2\right\}}$$

proof: The argument of the Bessel fcts is $\frac{\omega R}{c}\sin\vartheta = m\frac{\omega_0 R}{c}\sin\vartheta = m\frac{v}{c}\sin\vartheta$

and the Bessel fcts obey

$$J_{m-1}(x) - J_{m+1}(x) = 2 J_m'(x)$$

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x)$$

$$\rightarrow \underline{\frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - \frac{c^2}{v^2} J_m^2} = \frac{1}{2}\left[\left(\frac{m}{x}J_m - J_m'\right)^2 + \left(\frac{m}{x}J_m + J_m'\right)^2\right] - \frac{m^2 \sin^2\vartheta}{x^2} J_m^2$$

$$= \left(J_m'\right)^2 + \frac{m^2}{x^2} J_m^2 - \frac{m^2}{x^2}\sin^2\vartheta\, J_m^2 = \left(J_m'\right)^2 + \frac{m^2}{x^2}\left(1-\sin^2\vartheta\right) J_m^2$$

$$\overset{x=\frac{v}{c}m\sin\vartheta}{=} \left(J_m'\right)^2 + \frac{c^2}{v^2}\frac{1-\sin^2\vartheta}{\sin^2\vartheta} J_m^2 = \underline{\left(J_m'\right)^2 + \frac{c^2}{v^2}\frac{\cos^2\vartheta}{\sin^2\vartheta} J_m^2} \quad \square$$

discussion: (1) Angular integration yields the power emitted into the mth harmonic, $P_m = \int d\Omega\, dP_m/d\Omega$. The result is (see Problem 47)

Problem 46
Bessel fcts

$$\boxed{P_m = \frac{e^2}{R} m\omega_0 \left[2\beta^2 J_{2m}'(2m\beta) - (1-\beta^2)\int_0^{2m\beta} dx\, J_{2m}(x)\right]} \quad \beta = v/c$$

Problem 47
synchrotron radiation

An analysis (Problem 47) shows that P_m peaks at

$$m \approx m_c = \gamma^3$$

5/10/17

Week 6
Problem set 15 (#43, 44, 45)

$\rightarrow$ For ultrarelativistic particles, most of the power goes into very high frequencies $\rightarrow$ x-ray sources

(2) In the orbital plane, $\vartheta = \pi/2$, we have

$$\left.\frac{dP_m}{2\pi d\vartheta}\right|_{\vartheta=\pi/2} = \frac{\omega_0 e^2}{\pi R}\beta^3 m^2 \left(J_m'(\beta m)\right)^2 \approx \frac{\omega_0 e^2}{\pi R} m^2 \left(J_m'(m)\right)$$

and

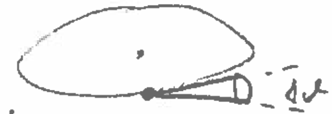
$$P_m = \frac{e^2}{R} m\omega_0\, 2\beta^2 J_{2m}'(2m\beta) \approx \frac{\omega_0 e^2}{R} 2m J_{2m}'(2m)$$

But $J_m'(m) \propto m^{-2/3}$ for $m \gg 1$

$$\rightarrow \frac{dP_m}{d\vartheta} \approx P_m\, m m^{-2/3} = P_m m^{1/3}$$

$$\to \qquad \frac{dP_m}{P_m} \simeq \frac{d\vartheta}{m^{-1/3}}$$

$\to$ The radiation is confined to a cone about $\vartheta = \pi/2$ of opening angle $\Delta\vartheta \sim m^{-1/3}$



That most of the radiation is emitted into harmonics around $m_c = \gamma^3$

$\to$ The cone opening is $\underline{\underline{\Delta\vartheta \propto 1/\gamma}}$ (see Problem 42 for a diff. argument that leads to the same condition)

At the ALS, $v/c = 0.999996$ $\to$ $\underline{\gamma = 350}$

6.3 Qualitative explanation of the main features

§6.2 $\to$ Synchrotron radiation is characterized by

(i) a narrow angle in the forward direction

(ii) high frequencies (high harmonics of the fundamental oscillation frequency)

These two characteristics can be qualitatively understood as follows:
For a point particle with trajectory $\vec{X}(t)$, we have the Liénard-Wiechert potential from Problem 35:

$$\vec{A}(\vec{x},t) = \frac{e\vec{v}(t_-)/c}{|\vec{x}-\vec{X}(t_-)| - \vec{v}(t_-)\cdot(\vec{x}-\vec{X}(t_-))/c} = \frac{e\vec{v}(t_-)/c}{|\vec{x}-\vec{X}(t_-)|}\,\frac{1}{1-\hat{n}\cdot\vec{v}(t_-)/c}$$

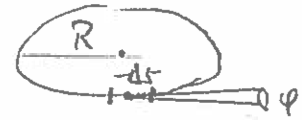
where $\quad t_- = t - \frac{1}{c}|\vec{x}-\vec{X}(t_-)|$

Let $\varphi = \sphericalangle(\vec{v},\hat{n})$ $\to$ $\underline{\dfrac{1}{1-\hat{n}\cdot\vec{v}/c}} = \dfrac{1}{1-\beta\cos\varphi} \overset{\beta\to 1}{\approx} \dfrac{1}{1-\beta(1-\frac{1}{2}\varphi^2)}$

$$\approx \frac{1}{\frac{1}{2}(1+\beta)(1-\beta)+\frac{1}{2}\varphi^2} = \frac{2}{1-\beta^2+\varphi^2}$$

$\to$ $\vec{A}$ is appreciable only for $\varphi \lesssim \sqrt{1-\beta^2} = 1/\gamma$. This explains (i)

Now consider a particle in a circular orbit.



The light reaches the observer only during a section Δs of the orbit given by $\frac{\Delta s}{2\pi R} \simeq \frac{\varphi}{2\pi} \rightsquigarrow \Delta s \simeq R\varphi$

$\rightsquigarrow$ The signal is emitted only during a time interval

$$\frac{\Delta t}{2\pi/\omega_0} \simeq \frac{\Delta s}{2\pi R} \simeq \frac{\varphi}{2\pi} \rightsquigarrow \Delta t \simeq \frac{1}{\omega_0}\varphi$$

$\rightsquigarrow$ The typical frequency emitted is

$$\omega_e \simeq \frac{1}{\Delta t} \simeq \omega_0/\varphi \simeq \omega_0\gamma \quad (*)$$

This holds in the rest frame of the particle. From the observer's point of view, Δs is shorter by a factor of $1/\gamma$ (Lorentz contraction) $\rightsquigarrow$ $\omega_e \propto \frac{1}{\Delta t} \propto \frac{1}{\Delta s}$ is larger by a factor of γ. Finally, the observer sees a Doppler shifted frequency (cf. ch. 3 §1.6), which provides another factor of γ

$$\rightsquigarrow \quad \omega_{\text{observed}} \simeq \overset{(*)}{\omega_0\gamma} \times \overset{\text{Lorentz contraction}}{\gamma} \times \underset{\text{Doppler effect}}{\gamma} = \omega_0\gamma^3$$

This explains (ii)

5/15/17

6.4 The polarization of synchrotron radiation

Polarization of light is measured via the effect of the $\vec{E}$-field.

$\rightsquigarrow$ Express the power spectrum in terms of $\vec{E}$ rather than $\vec{B}$.

$$\text{§4.3} \rightsquigarrow \frac{dW}{d\Omega} = \frac{c}{4\pi} r^2 \int \frac{d\omega}{2\pi}\, \hat{x} \cdot \left(\vec{E}(\vec{x},\omega) \times \vec{B}(\vec{x},-\omega)\right)$$

and §4.2 prop $\rightsquigarrow$ $\vec{E}(\vec{x},\omega) \simeq -\hat{x} \times \vec{B}(\vec{x},\omega) \rightsquigarrow \vec{B}(\vec{x},\omega) = \hat{x} \times \vec{E}(\vec{x},\omega)$

and (again §4.2 prop.) $\vec{B}(\vec{x},\omega) \propto \hat{x} \times \vec{j}(\vec{k},\omega)$

$$\to \vec{E}(\vec{x},\omega) \propto -\hat{x} \times (\hat{x} \times \vec{j}(\vec{k},\omega))$$

→ Our expressions for the power spectra remain valid if we replace $\hat{x} \times \vec{j} \to -\hat{x} \times (\hat{x} \times \vec{j})$

§5.1 →

$$\boxed{\frac{d^2P(T)}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau} \left[-\hat{x} \times (\hat{x} \times \vec{j}(\vec{k}, T+\tau/2))\right] \cdot \left[-\hat{x} \times (\hat{x} \times \vec{j}(\vec{k}, T-\tau/2))^*\right]}$$

def. 1: With a coordinate system chosen as in §6.2:

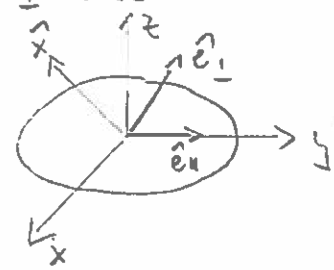
orbit in x-y plane, $\hat{x} = (\sin\theta, 0, \cos\theta)$,

we define parallel polarization as $\vec{E} \parallel \hat{e}_\parallel$ where

$$\hat{e}_\parallel = (0,1,0)$$

and perpendicular polarization as $\vec{E} \parallel \hat{e}_\perp$ where

$$\hat{e}_\perp = (-\cos\theta, 0, \sin\theta)$$



remark: (1) $\hat{e}_\parallel$ lies in the orbital plane and is $\perp$ to $\hat{x}$, $\hat{e}_\perp$ is $\perp$ to both $\hat{e}_\parallel$ and $\hat{x}$.

Now calculate the power radiated into the parallel polarization state:

$$\left(\frac{d^2P(T)}{d\omega\, d\Omega}\right)_\parallel = \frac{\omega^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau} \left[-\hat{x} \times (\hat{x} \times \vec{j}(\vec{k}, T+\tau/2))\right]_y \left[-\hat{x} \times (\hat{x} \times \vec{j}(\vec{k}, T-\tau/2))^*\right]_y$$

$[-\hat{x} \times (\hat{x} \times \vec{j})]_y = [\vec{j} - \hat{x}(\hat{x}\cdot\vec{j})]_y = j_y$ since $\hat{x} \perp \hat{y}$

, $\vec{j}(\vec{k},t) = e\vec{v}(t)\, e^{-ik\hat{x}\cdot\vec{y}(t)}$

$$- \frac{\omega^2 e^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau}\, e^{-i\lambda \hat{x}\cdot[\vec{\xi}(T+\tau/2)-\vec{\xi}(T-\tau/2)]}\, v_y(T+\tau/2)\, v_y(T-\tau/2)$$

lemma 1: $\boxed{v_y(T+\tau/2)\,v_y(T-\tau/2) = \frac{1}{2} v^2 (\cos 2\omega_0 T + \cos \omega_0 \tau)}$

proof: §6.2 $\rightarrow$ $v_y(T+\tau/2)\,v_y(T-\tau/2) = v^2 \cos(\omega_0 T + \omega_0\tau/2)\cos(\omega_0 T - \omega_0 \tau/2)$

$$= \frac{1}{2} v^2 [\cos 2\omega_0 T + \cos \omega_0 \tau]$$

lemma 2: $\boxed{\overline{e^{-i\lambda\hat{x}\cdot[\vec{\xi}(T+\tau/2)-\vec{\xi}(T-\tau/2)]}} = \sum_{m=-\infty}^{\infty} \left(J_m(\lambda R \sin\vartheta)\right)^2 e^{-im\omega_0\tau}}$

with $\overline{f(T)}$ = T-average over one oscillation period

proof: §6.2 lemma 3 $\leadsto$

$$\underline{\text{l.h.s.}} = \sum_{m\underline{m}} (-i)^m e^{-im\omega_0\tau/2} J_m(\lambda R\sin\vartheta)\,(+i)^{\underline{m}} e^{-i\underline{m}\omega_0\tau/2} J_{\underline{m}}(\lambda R \sin\vartheta) \times \underbrace{\overline{e^{i(m-\underline{m})\omega_0 T}}}_{=\delta_{m\underline{m}} \text{ by §6.2 lemma 4}}$$

$$= \underline{\text{r.h.s.}}$$

lemma 3: $\boxed{\overline{\cos 2\omega_0 T\, e^{-i\lambda\hat{x}[\vec{\xi}(T+\tau/2)-\vec{\xi}(T-\tau/2)]}} = -\sum_{m=-\infty}^{\infty} J_{m+1}(\lambda R\sin\vartheta)\, J_{m-1}(\lambda R \sin\vartheta) \times e^{-im\omega_0\tau}}$

proof: $\underline{\text{l.h.s.}} = \sum_{m\underline{m}} (-i)^m (+i)^{\underline{m}} J_m J_{\underline{m}}\, \overline{e^{-i(m-\underline{m})\omega_0 T} \frac{1}{2}\left(e^{2i\omega_0 T} + e^{-2i\omega_0 T}\right)}\, e^{-i(m+\underline{m})\frac{\omega_0\tau}{2}}$

$$= \sum_{m\underline{m}} (-i)^m (+i)^{\underline{m}} J_m J_{\underline{m}}\, \frac{1}{2}\left(\delta_{\underline{m},m+2} + \delta_{\underline{m},m-2}\right) e^{-i(m+\underline{m})\omega_0\tau/2}$$

$$= -\frac{1}{2}\sum_m \left(J_m J_{m+2}\, e^{-i(m+1)\omega_0\tau} + J_m J_{m-2}\, e^{-i(m-1)\omega_0\tau}\right)$$

$$= -\frac{1}{2}\sum_m \left(J_{m-1}J_{m+1} + J_{m+1}J_{m-1}\right) e^{-im\omega_0\tau} = \underline{\text{r.h.s.}}$$

$$\rightarrow \boxed{\left(\frac{d^2 \mathcal{P}(\tau)}{d\omega\, d\Omega}\right)_{\parallel}} = \frac{\omega^2 e^2}{4\pi^2 c^3} \int d\tau\, e^{i\omega\tau}\, \frac{1}{2} v^2 \sum_{m=-\infty}^{\infty} \left[\cos \omega_0 \tau\, J_m^2 - J_{m+1} J_{m-1}\right] e^{-i m \omega_0 \tau}$$

$$= \frac{\omega^2 e^2}{4\pi^2 c^3}\, \frac{1}{2} v^2 \sum_{m=-\infty}^{\infty} \left[\frac{1}{2} J_{m+1}^2 + \frac{1}{2} J_{m-1}^2 - J_{m+1} J_{m-1}\right] \delta(\omega - m\omega_0)$$

$\rightarrow \left(\frac{d\mathcal{P}_m}{d\Omega}\right)_{\parallel}$ is given by the expression for §6.2 with

$$\frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - \frac{c^2}{v^2} J_m^2$$

replaced by

$$\frac{1}{2}\left(J_{m+1}^2 + J_{m-1}^2\right) - J_{m+1} J_{m-1} = \frac{1}{2}\left(J_{m+1} - J_{m-1}\right)^2 = 2\left(J_m'\right)^2$$

theorem: The power radiated into the mth harmonic with parallel polarization is given by

$$\boxed{\left(\frac{d\mathcal{P}_m}{d\Omega}\right)_{\parallel} = \frac{\omega_0 e^2}{8\pi R}\left(\frac{v}{c}\right)^3 m^2 \left[J_m'\left(m \frac{v}{c} \sin\vartheta\right)\right]^2}$$

remark: (2) This is the first of the two terms in the expression for $d\mathcal{P}_m/d\Omega$ on p 96

corollary: The power radiated into the mth harmonic with perpendicular polarization is given by the second term on p 96:

$$\boxed{\left(\frac{d\mathcal{P}_m}{d\Omega}\right)_{\perp} = \frac{\omega_0 e^2}{8\pi R}\left(\frac{v}{c}\right)^3 m^2 \left(\frac{J_m\left(m \frac{v}{c} \sin\vartheta\right)}{\frac{v}{c} \tan\vartheta}\right)^2}$$

Week 7
Problem 16 (#46,47 pt I)

Plot the two contributions :

In[44]:=
```
vc = 0.99
gamma = 1 / (1 - vc^2) ^ (1 / 2)
gamma^3
m = Floor[gamma^3]
J[m_, x_] := BesselJ[m, x]
JPrime[m_, x_] := (BesselJ[m - 1, x] - BesselJ[m + 1, x]) / 2
fpar[theta_] := m^2 (JPrime[m, m vc Sin[theta]]) ^2
Plot[fpar[x], {x, 0, Pi}, PlotRange → All]
fperp[theta_] := m^2 (J[m, m vc Sin[theta]] / (vc Tan[theta])) ^2
Plot[fperp[x], {x, 0, Pi}, PlotRange → All]
Plot[fpar[x] + fperp[x], {x, 0, Pi}, PlotRange → All]
```

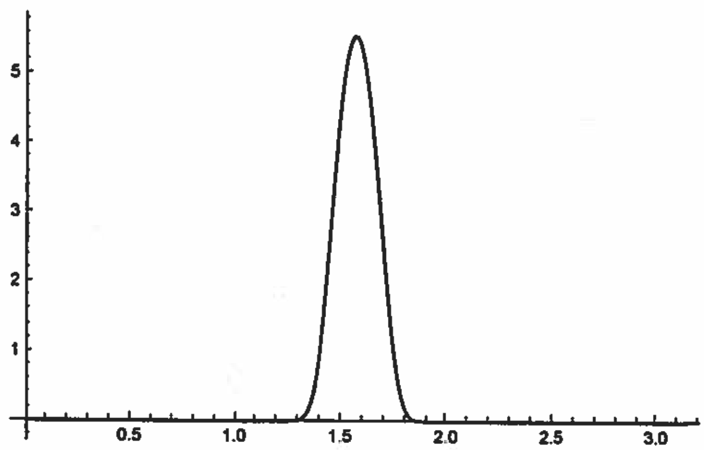
Out[44]= 0.99

Out[45]= 7.08881

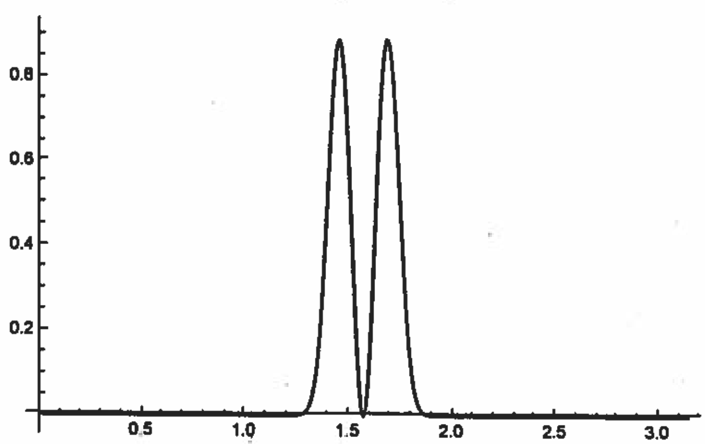
Out[46]= 356.222

Out[47]= 356

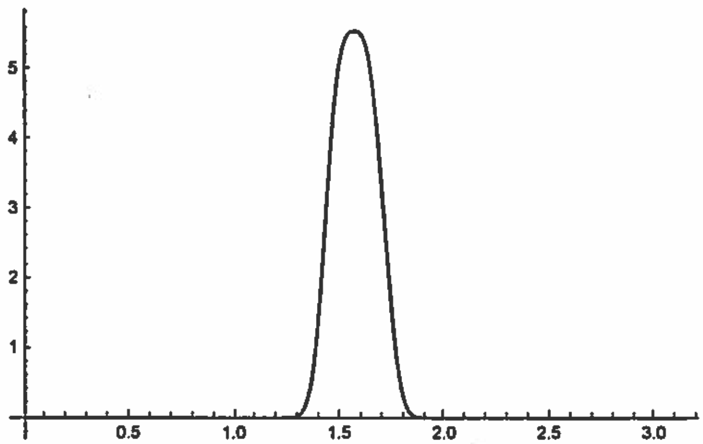
Out[51]=



Out[53]=



Out[54]=



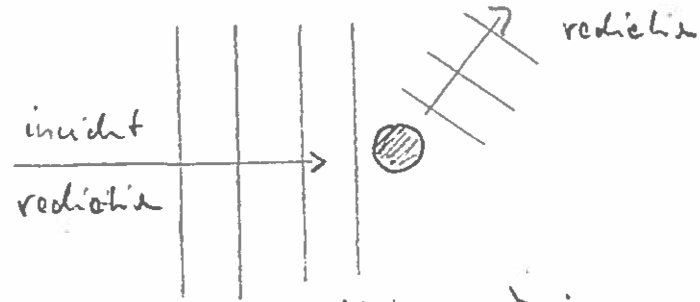
remark: (3) $(dP_\omega/d\Omega)_{\parallel,\perp}$ have a maximum and minimum, respectively at $\theta = \pi/2$! This is a tell-tale signature of synchrotron radiation that is important for astrophysical observations

Week 7

Problem set 16 (# 46, 47 pt I)

§7 Scattering of light by small obstacles

7.1 The scattering cross section



Consider a charged object in an electromagnetic field.

The field accelerates the object, which then radiates. This physical situation is referred to as scattering if the size of the object is small compared to the wavelength of the radiation.

Let P_{scatt} be the power radiated by the accelerated object.

Compare this to the power of the e.m. field

ch 1 §3.6 $\rightarrow$ the Poynting vector $\vec{P} = \frac{c}{4\pi} \vec{E} \times \vec{B}$ is the energy current density of the field, i.e. the power per unit area.

def.: The quantity $\boxed{\sigma := P_{scatt} / |\vec{P}|}$

is called scattering cross section

remark: (1) dimensionally, σ is an area

(2) σ is a measure of the object's effective cross-sectional area seen by the incident radiation.

7.2 Thomson scattering

def.: Scattering of electromagnetic radiation by free charges is called Thomson scattering (J.J. Thomson 1856-1940, Nobel Prize 1906) consider a nonrelativistic particle with charge e, mass m:

$$\to \quad m\dot{\vec v} = e\vec E$$

P_{scatt} is given by the Larmor formula, ch 4 §3.3 prop.:

$$\underline{P_{scatt}} = \frac{2e^2}{3c^3}(\dot{\vec v})^2 = \frac{2e^2}{3c^3}\left(\frac{e}{m}\right)^2 \vec E^2 = \underline{\frac{2e^4}{3m^2c^3}\vec E^2}$$

The Poynting vector in vacuum is (ch 3 §1.3)

$$\underline{\vec P} = cu\hat n = \frac{c}{8\pi}(\vec E^2+\vec B^2)\hat n = \underline{\frac{c}{4\pi}\vec E^2\hat n} \quad \to \quad \underline{|\vec P| = \frac{c}{4\pi}\vec E^2}$$

proposition 1: The cross section for Thomson scattering is

$$\boxed{\sigma = \frac{8\pi e^4}{3m^2c^4}}$$

Thomson cross section

remark: (1) For electrons, $e^2/mc^2 = r_e$ is the classical electron radius, see ch 3 §2.5.

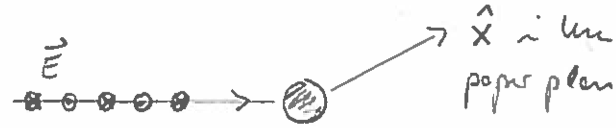
$$\to \quad \sigma = \frac{8\pi}{3}r_e^2 = 0.66\times 10^{-24}\ \text{cm}^2$$

consider the angular distribution of the scattered radiation.

ch 4 §3.3 $\to$

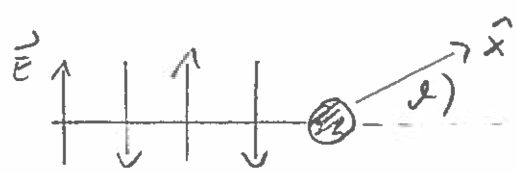
$$\underline{\frac{dP_{scatt}}{d\Omega}} = \frac{e^2}{4\pi c^3}\left[\dot{\vec v}^2 - (\hat x\cdot\dot{\vec v})^2\right] = \underline{\frac{e^4}{4\pi m^2c^3}\vec E^2\left[1-(\hat x\cdot\hat E)^2\right]}$$

1st case: $\vec E \perp \hat x \quad \to \quad \hat E\cdot\hat n = 0$



$$\to \quad 1-(\hat x\cdot\hat E)^2 = 1$$

2nd case: $\vec E$ in the same plane as $\hat x$

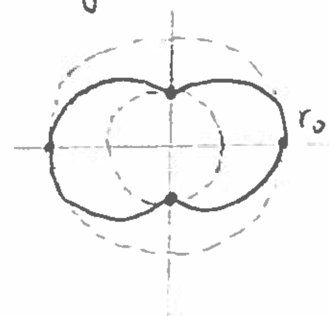


$\to |\hat E\cdot\hat x| = \sin\theta \to 1-(\hat x\cdot\hat E)^2 = \cos^2\theta$

remark: (2) In the 2nd case, there is no scattering for $\vartheta = \pm\frac{\pi}{2}$, consistent with the fact that no radiation is emitted in the direction of the acceleration.

proposition 2: For unpolarized incident radiation, the Thomson scattering cross section per solid angle is

$$\boxed{\frac{d\sigma}{d\Omega} = r_0^2 \, \frac{1+\cos^2\vartheta}{2}}$$



with $r_0 = e^2/mc^2$

proof: unpolarized radiation $\rightsquigarrow$ average over the two cases

$$\to \quad \overline{1-(\hat{x}\cdot\hat{E})^2} = \frac{1}{2}(1+\cos^2\vartheta)$$

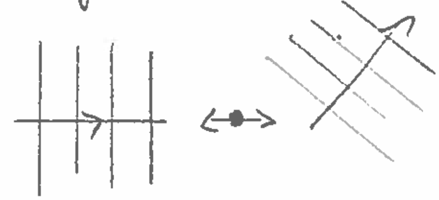
$$\to \quad \frac{d\sigma}{d\Omega} = \frac{dP_{scatt}/d\Omega}{|\vec{P}|} = \frac{e^4 \cancel{\vec{E}^2}}{4\pi m^2 c^3}\,\frac{1}{2}(1+\cos^2\vartheta)\,\frac{4\pi}{c}\,\frac{1}{\cancel{\vec{E}^2}} = \frac{e^4}{m^2c^4}\,\frac{1}{2}(1+\cos$$

check: $\int d\Omega \frac{d\sigma}{d\Omega} = r_0^2 \frac{1}{2} 2\pi \int_{-1}^{1} d\eta\,(1+\eta^2) = r_0^2 2\pi \left(1+\frac{1}{3}\right) = \frac{8\pi}{3} r_0^2$ ✓

remark: (3) Thomson scattering is important in astrophysics (solar corona, linear polarization of CMB) and plasma physics (for measuring the electron temperature and density in a plasma).

7.3 Scattering by a bound charge

Now consider the scattering of light by a bound charge e, modeled as a harmonic oscillator with mass m and resonance frequency ω_0 and damping constant γ.



proposition: The scattering cross section in this case is

$$\boxed{\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

remark: (1) For $\omega \gg \omega_0, \gamma$ this reduces to the Thomson value from §7.1, as it must.

proof: The eq. of motion for the charge is

$$m\ddot{\vec{x}} + m\omega_0^2\vec{x} + m\gamma\dot{\vec{x}} = e\vec{E}$$

with $\vec{x}(t)$ the position of the charge and $\vec{E}$ taken at the equilibrium position of the harm. osc.

remark: (2) This is a harm. osc. driven by the external force $e\vec{E}(t)$, and we assume the oscillator amplitude to be small compared to λ.

Temporal Fourier trafo $\to -m\omega^2\vec{x}(\omega) + m\omega_0^2\vec{x}(\omega) + i\omega m\gamma\vec{x}(\omega) = e\vec{E}(\omega)$

$$\to \quad \vec{x}(\omega) = \frac{(e/m)\vec{E}(\omega)}{\omega_0^2-\omega^2+i\gamma\omega}$$

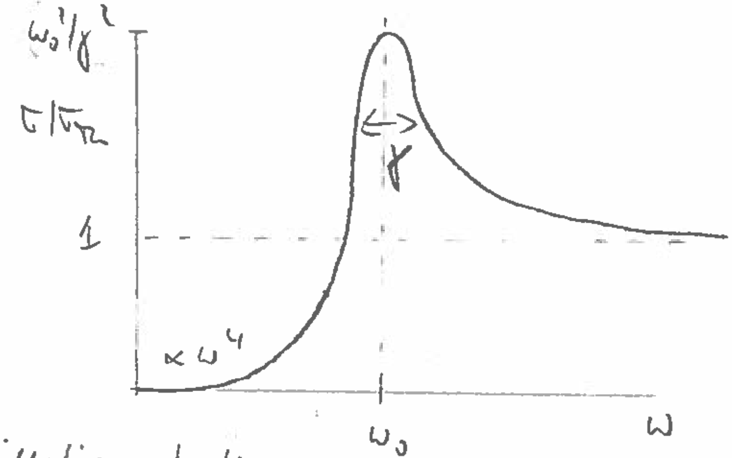
$$\to \quad |\dot{\vec{v}}(\omega)|^2 = |\omega^2\vec{x}(\omega)|^2 = \frac{e^2}{m^2}|\vec{E}(\omega)|^2 \underbrace{\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}_{=:\,\chi(\omega)}$$

Now we use the Larmor formula again:

$$P_{scatt} = \frac{2e^2}{3c^3}|\dot{\vec{v}}|^2 = \frac{2e^2}{3c^3}\frac{e^2}{m^2}|\vec{E}|^2\chi(\omega)$$

and $|\vec{P}|^2 = \frac{c}{4\pi}|\vec{E}|^2$ (see §7.2)

$$\leadsto \underline{\sigma} = P_{scatt}/|\vec{P}| = \frac{2e^4}{3c^3m^2}\frac{4\pi}{c}\chi(\omega) = \underline{\underline{\frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2\chi(\omega)}}$$



discussion: (0) Note that this entire discussion is closely related to §4.5

(1) $\sigma(\omega)$ shows a resonance at $\omega = \omega_0$ with width γ.

(2) For $\omega \gg \omega_0$, the fast oscillation of the force leaves the harm. osc. no time to respond $\to \sigma \to \sigma_{Th}$ (see remark (ii))

(3) For $\omega \ll \omega_0$, $\sigma \propto \omega^4$ ("Rayleigh scattering"). This is why the sky is so blue in Alesene.

(4) We have considered γ fixed. However, there is damping due to the radiation itself, see §4.5. This is the problem of "radiation damping", see Jackson Sec. 17 and Schwinger et al Sec. 45.4

Problem 48

scattering by dielectric sphere

5/22/17