

## Problem Assignment # 2

01/15/2025  
due 01/22/2025**0.2.4. Functional derivative** (3 pts)

Let  $F[\varphi]$  be a functional of a real-valued function  $\varphi(x)$ . For simplicity, let  $x \in \mathbb{R}$ ; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of  $F$  as

$$\frac{\delta F}{\delta \varphi(x)} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left( F[\varphi(y) + \epsilon \delta(y-x)] - F[\varphi(y)] \right)$$

a) Calculate  $\delta F/\delta \varphi(x)$  for the following functionals:

i)  $F = \int dx \varphi(x)$

ii)  $F = \int dx \varphi^2(x)$

iii)  $F = \int dx f(\varphi(x)) g(\varphi(x))$  where  $f$  and  $g$  are given functions

iv)  $F = \int dx (\varphi'(x))^2$  where  $\varphi'(x) = d\varphi/dx$

*hint:* Integrate by parts and assume that the boundary terms vanish.

v)  $F = \int dx V(\varphi'(x))$  where  $V$  is some given function.

*remark:* Blindly ignore terms that formally vanish as  $\epsilon \rightarrow 0$  unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density  $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$  and an action  $S = \int d^4x \mathcal{L}$ . Show that extremizing  $S$  by requiring  $\delta S/\delta \varphi(x) \equiv 0$  with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

**0.2.5. Massive scalar field** (2 pts)

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field  $\phi(x) \in \mathbb{C}$ :

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi(x)) (\partial^\mu \phi^*(x)) - \frac{m^2}{2} |\phi(x)|^2$$

with  $\phi^*$  the complex conjugate of  $\phi$ . What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation,  $\phi(x) \rightarrow \phi(x) e^{i\Lambda(x)}$ , with  $\Lambda(x)$  a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

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### 0.3.1. Particle in homogeneous $\mathbf{E}$ and $\mathbf{B}$ fields (6 pts)

Consider a point particle (mass  $m$ , charge  $e$ ) in homogeneous fields  $\mathbf{B} = (0, 0, B)$  and  $\mathbf{E} = (0, E_y, E_z)$ . Treat the motion of the particle nonrelativistically.

a) Show that the motion in  $z$ -direction decouples from the motion in the  $x$ - $y$  plane, and find  $z(t)$ .

b) Consider  $\xi := x + iy$ . Find the equation of motion for  $\xi$ , and its most general solution.

*hint:* Define the *cyclotron frequency*  $\omega = eB/mc$ , and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by  $\mathbf{B}$  and  $\mathbf{E}$  is given by the *drift velocity*

$$\langle \mathbf{v} \rangle = c \mathbf{E} \times \mathbf{B} / B^2$$

Show that  $E_y/B \ll 1$  is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the  $x$ - $y$  plane can have three qualitatively different shapes, and plot a representative example for each.

### 0.3.2. Harmonic oscillator coupled to a magnetic field (4 pts)

Consider a charged 3-d classical harmonic oscillator (oscillator frequency  $\omega_0$ , charge  $e$ ). Put the oscillator in a homogeneous time-independent magnetic field  $\mathbf{B} = (0, 0, B)$ . Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to  $\mathbf{B}$ .