PHYS 622 | Classical Electrodynamics | W 2025

Problem Assignment # 5 / Midterm | 02/05/2025
due 02/12/2025

This assignment doubles as the Midterm. You can use any inanimate resource you like, but please don't consult any live resources other than me. It is due on February 5, 2025 at 2pm (start of class). Submit to me, **not** to your grader, either electronically or in hard copy.

1.3.1. **Magnetic Monopoles** (8 pts)

In ch. 1 we noticed that the Maxwell equations are weirdly asymmetrical. This can be 'fixed' as follows. Suppose nature had decided that one 4-vector potential A^μ and one 4-current J^μ was not enough, and there was another 4-vector field $\tilde{A}^\mu$ and another 4-current $\tilde{J}^\mu$. Now define a modified field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\kappa \epsilon_{\mu\nu}{}^{\kappa\lambda} \tilde{A}_\lambda$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the 4-dimensional Levi-Civita symbol, and a modified Lagrangian density

$$\mathcal{L} = \frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_\mu J^\mu - \frac{1}{c} \tilde{A}_\mu \tilde{J}^\mu$$

a) Show that the resulting Euler-Lagrange equations have the form

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \qquad , \qquad \partial_\mu \tilde{F}^{\mu\nu} = \frac{4\pi}{c} \tilde{J}^\nu$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual field tensor from Problem 1.1.1. Compare these equations of motion with the corresponding ones in Maxwell theory.

b) Define electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ by parameterizing the modified field tensor $F^{\mu\nu}$ as in Maxwell theory, express the equations of motion from part a) in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$, and compare the result with the usual Maxwell equations.

c) Show that the 4-current $\tilde{J}^\mu$ obeys a continuity equation and briefly discuss its physical meaning.

0.3.4 Relativistic Kepler problem (12 pts)

Consider the motion of a point mass m in an attractive Coulomb potential centered at the origin within the framework of special relativity:

$$L = m\,c^2 - m\,c^2 \sqrt{1 - (\vec{v}/c)^2} + \alpha/r \quad , \quad (\alpha > 0\ ,\ r = |\vec{x}|) \quad .$$

Let the z-component of the angular momentum be $\ell \geq 0$.

a) Use the conservation laws for the energy and the angular momentum to find the equation for the sectorial velocity

$$r^2\,\dot{\phi} = g(r; E, \ell)$$

where ϕ is the azimuthal angle, and the radial equation of motion

$$\dot{r}^2 = c^2 f(r; E, \ell)$$

Determine the functions g and f explicitly and show that in the nonrelativistic limit they correctly reduce to the Galilean case.

hint: Take the angular momentum $\vec{\ell}$ to point in the z-direction. You know from Mechanics that the rotational invariance of L implies that the orbit lies in a plane perpendicular to $\vec{\ell}$; you can use that without proof. Use polar coordinates in the orbital plane.

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b) Assume $\ell = 0$, and let $\dot{r}(t = 0) = 0$, $r(t = 0) = 2a$. Discuss and draw $\dot{r}$ as a function of r, and compare with the Galilean case. Determine the oscillation period $T(a)$ in terms of a dimensionless integral that depends only on the parameter $\xi = \alpha/2amc^2$. Discuss your result in the nonrelativistic and ultrarelativistic limits ($\xi \to 0$ and $\xi \to \infty$, respectively). Show that Kepler's third law gets modified within special relativity, and plot $T/T_{\xi=0}$ as a function of ξ.

hint: To see why the initial condition $r(t = 0) = 2a$ is sensible, consider the apocenter for a Keplerian elliptical orbit in the limit $\ell \to 0$. Define the orbital period as the time between successive apocenters (which for any Keplerian orbit with $\ell > 0$ corresponds to one complete revolution) and note the qualitative difference between an orbit with $\ell = 0$ and one with positive ℓ, no matter how small. Check your result in the nonrelativistic limit, including the prefactor, against the standard Keplerian one (e.g., Landau & Lifshitz Vol. 1 §15.).

Note: Remarkably, the problem can be solved in closed form even for $\ell > 0$. If you are interested, have a look at the unabbreviated version of Problem 0.3.4 as it is formulated in the Lecture Notes.