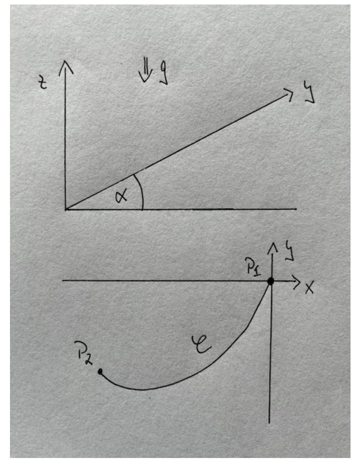
PHYS 622 | Classical Electrodynamics | W 2025

Problem Assignment # 1 | 01/08/2025
due 01/15/2025

0.2.1 **The brachistochrone problem** (18 pts)

A point mass glides without friction on an inclined plane (inclination angle α) from point P_1 to point P_2 on a path $\mathfrak{C}$ according to Galilean mechanics.



a) Use energy conservation to find the velocity as a function of y, using the coordinate system in the sketch.

b) Write the passage time from P_1 to P_2 in the form

$$T = \int_{x_1}^{x_2} dx\, L(y, y')$$

with y considered a function of x and $y' = dy/dx$, and determine the Lagrangian L. Use the fact that Jacobi's integral is constant to find an ODE for y.

c) Substitute $y' = \cot t$ to write the brachistochrone, i.e., the solution of the ODE from part b), in a parametric form, $y = y(t)$, $x = x(t)$.

d) Express the passage time as a function of the value t_2 of the brachistochrone parameter in the point P_2 (or, equivalently, as a function of $y'_2 = (dy/dx)_{\mathrm{P}_2}$, which has a more intuitive meaning).

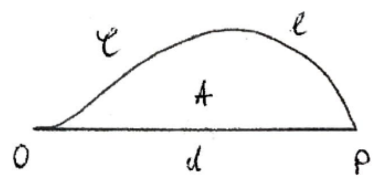
e) Find the passage time for the shortest path from P_1 to P_2 (as opposed to the brachistochrone) as a function of t_2.

f) Discuss the ratio of the two passage times as a function of t_2.

hint: The parameter value t_2 for the brachistochrone at the end point P_2 is a known function of y'_2. It therefore suffices to discuss the passage time as a function of t_2.

0.2.2 **Dido's problem** (6 pts)

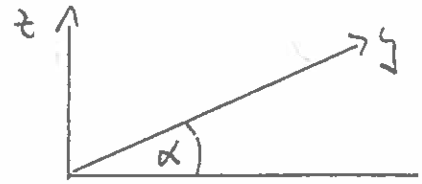
An area A in the x-y-plane is enclosed by a straight line between two points O and P that are a distance d apart, and a path $\mathfrak{C}$ with end points O and P and length $\ell > d$. Find the path $\mathfrak{C}$ that maximizes A.

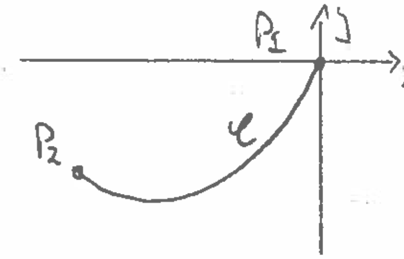


0.2.3 **Geodesics on the 2-sphere** (5 pts)

Show that the geodesics on the 2-sphere are great circles.

hint: There are various ways of doing this. One is to set up the problem of geodesics in $\mathbb{R}_3$ with the constraint that the desired paths $\vec{x}(t)$ must lie on the sphere. Now use the Euler-Lagrange equations for the constrained problem to show that $\vec{\ell} = \vec{x} \times \vec{p} = \text{const}$, where $\vec{p} = \partial L/\partial \dot{\vec{x}}$, with L the appropriate Lagrangian.





0.2.1.) constraint: $z = y \sin\alpha$

let $P_1 = (0,0,0)$ wlog

a) Energy conservation ->

$$\frac{m}{2} v^2 = -mgz = -mgy\sin\alpha$$

(1) -> $\underline{v(x,y)} = v(y) = \sqrt{2g\sin\alpha}\sqrt{-y} =: \underline{\sqrt{-ay}}$ where $\underline{a := 2g\sin\alpha}$

b) length of infinitesimal path segment: $dl = dt\sqrt{\dot{x}^2+\dot{y}^2}$

(1) Time needed to move across dl: $dT = dl/v(y)$

-> Passage time

$$\underline{T} = \int_{t_1}^{t_2} dt \, \frac{1}{v(y)} \sqrt{\dot{x}^2+\dot{y}^2} = \int_{x_1}^{x_2} dx \, \frac{1}{v(y)} \sqrt{1+y'^2}$$

(1)
$$= \underline{\int_{x_1}^{x_2} dx \, L(y,y')} \quad \text{with} \quad \underline{L(y,y') = \frac{1}{v(y)}\sqrt{1+y'^2}}$$

ch 0 §2.3 remark (3)(ii) -> Jacobi's integral is a constant of motion, i.e.,

$$H(y,y') = y'\frac{\partial L}{\partial y'} - L = \frac{y'^2}{v(y)\sqrt{1+y'^2}} - \frac{1}{v(y)}\sqrt{1+y'^2} = \frac{-1}{v(y)\sqrt{1+y'^2}} = \text{const}$$

(1) -> $\underline{y(1+y'^2) = \text{const} =: c_1 < 0}$

c) substitute $y' = \cot t$, $t = \operatorname{arccot} y'$

(1) $\rightsquigarrow \underline{y} = \frac{c_1}{1+y'^2} = \frac{c_1}{1+\cot^2 t} = c_1 \sin^2 t = \underline{\frac{1}{2} c_1 (1-\cos 2t)}$

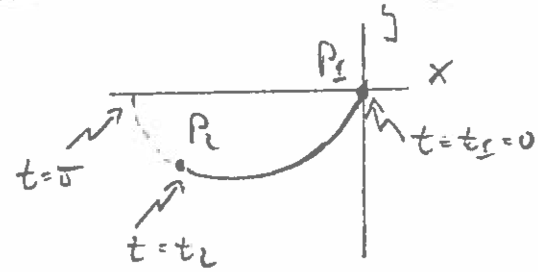
$dx = \frac{dy}{y'} = \frac{2 c_1 \sin t \cos t \, dt}{\cot t} = 2 c_1 \sin^2 t \, dt = c_1 (1-\cos 2t)\, dt$

(1) $\rightsquigarrow \underline{x} = c_2 + c_1 t - \frac{1}{2} c_1 \sin 2t = \underline{\frac{1}{2} c_1 (2t - \sin 2t) + c_2}$

initial conditions: $y_1 = y(t=0) = 0$ ✓

$x_1 = x(t=0) = c_2 \overset{!}{=} 0 \rightsquigarrow \underline{c_2 = 0}$

(1) let $c = \frac{1}{2} c_1 \rightsquigarrow$ $\boxed{\begin{aligned} x(t) &= c(2t - \sin 2t) \\ y(t) &= c(1-\cos 2t) \end{aligned}}$ Brachistochrone in parametric form



remark: (1) This is for $x_2 < 0$
If $x_2 > 0$, consider $t < 0$

(2) c is a scale factor that must be chosen such that P_2 lies on the curve.

To bring this in standard form, let $\tau = 2t - \pi$, $\hat{x} = -x + \pi c$

$\rightsquigarrow \underline{\hat{x}(\tau)} = -c\left(\tau + \cancel{\pi} - \sin(\tau+\pi)\right) + \cancel{\pi c} = \underline{-c(\tau + \sin\tau)}$

(1) $\underline{y(\tau)} = c\left(1 - \cos(\tau+\pi)\right) = c(1+\cos\tau) = \underline{-c(-1-\cos\tau)}$ $\Big\}$ cycloid

d) Part c) =>

$$\boxed{\begin{array}{l} x(t) = c\,(2t - \sin 2t) \\ y(t) = c\,(1-\cos 2t) \end{array}}$$

$c<0$ $\qquad$ $t = \operatorname{arc\,cot} y'$ $\qquad$ $y' = dy/dx$

$\rightsquigarrow$ $\dot x(t) = 2c\,(1-\cos 2t)$

$\dot y(t) = 2c \sin 2t$

$\rightsquigarrow$ $\dot x^2 + \dot y^2 = 4c^2(1-\cos 2t)^2 + 4c^2\sin^2 2t = 4c^2 - 8c^2\cos 2t + 4c^2 = 8c^2(1-\cos 2t)$

1 $\qquad = 16c^2 \sin^2 t$

$\rightsquigarrow$ passage time for the brachistochrone

$$T_b = \int_0^{t_2} dt\, \frac{1}{\sqrt{-ay}}(-4c)\sin t = \frac{-4c}{\sqrt{-ac}}\int_0^{t_2} dt\, \frac{1}{\sqrt{1-\cos 2t}}\sin t$$

$$= \frac{-4c}{\sqrt{-ac}}\int_0^{t_2} dt\, \frac{\sin t}{\sqrt{2}\sin t} = \frac{-\sqrt{2}c}{\sqrt{-ac}}\, 2\int_0^{t_2} dt$$

1 $\qquad = 2\sqrt{-2c/a}\; t_2$ $\qquad$ with $\qquad t_2 = \operatorname{arc\,cot} y'_2$

e) straight line: $\qquad x(\tau) = x_2\tau$, $\quad y(\tau) = y_2\tau$ $\qquad \tau_1 = 0,\ \tau_2 = 1$ $\qquad (y_2<0)$

$\rightsquigarrow$ passage time for straight line

1 $$T_s = \int_0^1 d\tau\, \frac{1}{\sqrt{-a y_2 \tau}}\sqrt{x_2^2+y_2^2} = \frac{\sqrt{x_2^2+y_2^2}}{\sqrt{-a y_2}}\int_0^1 d\tau\, \tau^{-1/2} = \frac{2}{\sqrt{-a y_2}}\sqrt{x_2^2+y_2^2}$$

But we know that the point (x_2, y_2) lies on the brachistochrone

1 $\quad$ c) $\rightsquigarrow$ $\qquad x_2 = c\,(2t_2 - \sin 2t_2)$, $\quad y_2 = c\,(1-\cos 2t_2)$

$\rightsquigarrow$ $x_2^2 + y_2^2 = c^2\left(4t_2^2 - 4t_2\sin 2t_2 + \cancel{\sin^2} 2t_2 + 1 - 2\cos 2t_2 + \cancel{\cos^2} 2t_2\right)$

$= c^2\left(2(1-\cos 2t_2) + 4t_2^2 - 4t_2 \sin 2t_2\right)$

$= c^2\left(4\sin^2 t_2 + 4t_2^2 - 4t_2\sin 2t_2\right)$

1 $$= 4c^2\left(\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2\right)$$

$$\rightsquigarrow \quad T_s = \frac{2}{\sqrt{-gc}} \frac{1}{\sqrt{1-\cos 2t_2}} (-2c) \sqrt{\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2}$$

$$= \frac{+2\cdot\cancel{2}c}{\cancel{\sqrt{-gc}}} \frac{1}{\cancel{\sqrt{2}}\sin t_2} \sqrt{\sin^2 t_2 + t_2^2 - t_2 \sin 2t_2}$$

1 $$= \sqrt{-2c/g}\ \frac{+2}{\sin t_2} \sqrt{t_2^2 + \sin^2 t_2 - t_2 \sin 2t_2}$$ with t_2 from d)

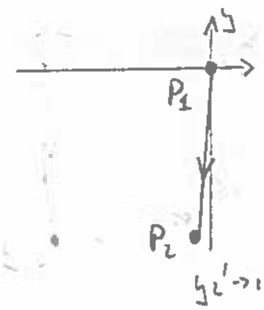
f) $t_2 \to 0$ $\quad (\rightsquigarrow y_2' \to \infty)$

$$T_s(t_2 \to 0) = \sqrt{-2c/g}\ \frac{-2}{t_2(1+O(t_2^2))} \sqrt{\cancel{t_2^2} + \cancel{t_2^2} - \frac{2}{6}t_2^4 - \cancel{2t_2^2} + \frac{8}{6}t_2^4 + O(t_2^6)}$$

$$= -2\sqrt{-2c/g}\ t_2 \left[1+O(t_2^2)\right]$$

1 $$= T_b\left[1+O(t_2^2)\right]$$

This is the case when the end point P_2 lies close to the fall line.

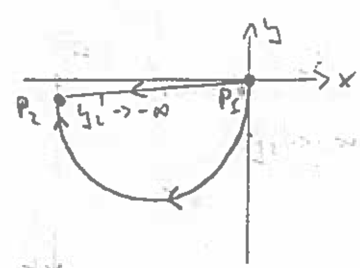


1 $\rightsquigarrow$ $y_2' \to \infty$, $t_2 \to 0$, and brachistochrone and straight line are almost identical.

$t_2 = -\pi - \varepsilon$ $\quad (\rightsquigarrow y_2' \to -\infty)$

1 $$T_s(t_2 = -\pi - \varepsilon) = \sqrt{-2c/g}\ \frac{2\pi}{\varepsilon}\left[1+O(\varepsilon^2)\right] = T_b(t_2 = -\pi)\ \frac{1}{\varepsilon}\left[1+O(\varepsilon^2)\right]$$

This is the case when P_2 lies close to the x-axis $\rightsquigarrow$ $y_2' \to -\infty$, and the straight-line path is very slow.



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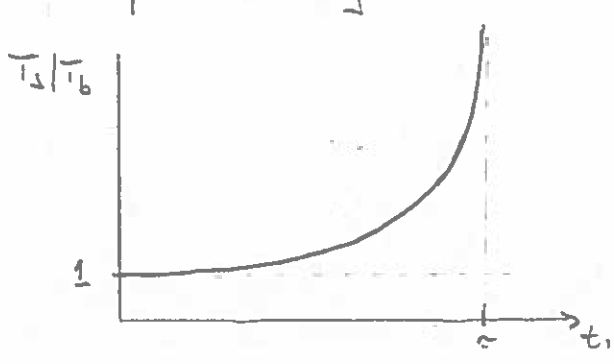


0.2.2.) cf. §2.4 ex (5t) $\rightsquigarrow$ $A = \frac{1}{2} \oint dt \, [x(t)\dot{y}(t) - y(t)\dot{x}(t)]$

$$\ell = \oint dt \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

(1) cf. §2.4 thm $\rightsquigarrow$ $L = \frac{1}{2}(x\dot{y} - y\dot{x}) + \frac{1}{2}\lambda\sqrt{\dot{x}^2 + \dot{y}^2}$

EL eqs: $\frac{1}{2}\dot{y} = \frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\frac{1}{2}\dot{y} + \frac{1}{2}\lambda \frac{d}{dt}\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$

$$\rightsquigarrow \quad y - y_0 = \lambda \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \qquad (1)$$

$$-\frac{1}{2}\dot{x} = \frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}\dot{x} + \frac{1}{2}\lambda \frac{d}{dt}\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

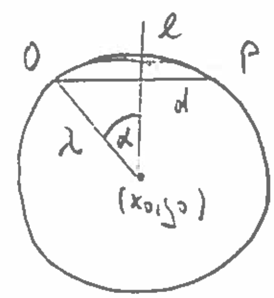
(1)

$$\rightsquigarrow \quad x - x_0 = -\lambda \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \qquad (2)$$

(1) $\rightsquigarrow$ $\boxed{(x-x_0)^2 + (y-y_0)^2 = \lambda^2}$ circle with center (x_0, y_0), radius λ

Now $\overline{OP} = d = 2\lambda \sin\alpha$

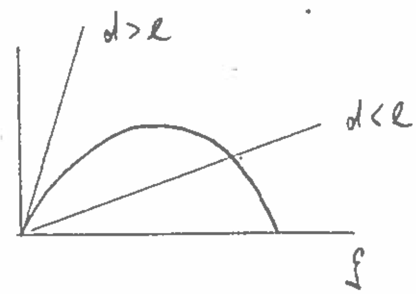
and $\ell = 2\lambda\alpha$,



(1) $\rightsquigarrow$ $\boxed{\frac{d}{2\lambda} = \sin\frac{\ell}{2\lambda}}$ determines λ

Define $\xi := \ell/2\lambda$

(1) $\rightsquigarrow$ $\boxed{\sin\xi = \frac{d}{\ell}\xi}$.



Graphic solution $\rightsquigarrow$

1st case: $d \geq \ell$ no solution

$d < \ell$ exactly one solution, which obviously is a maximum

(1)

0.2.3.) Consider a path in $\mathbb{R}_3$: $\vec{x}(t) = (x_1(t), x_2(t), x_3(t))$

length of curve: $l = \int_{t_-}^{t_+} dt \sqrt{\dot{x}_1^2(t) + \dot{x}_2^2(t) + \dot{x}_3^2(t)}$

$$=: \int_{t_-}^{t_+} dt \; L_1(\dot{\vec{x}})$$

1

Boundary condition: $x_1^2(t) + x_2^2(t) + x_3^2(t) = 1 \quad (+)$

or $0 = \int_{t_-}^{t_+} dt \left(x_1^2(t) + x_2^2(t) + x_3^2(t) - 1\right)$

$$=: \int_{t_-}^{t_+} dt \; L_2(\vec{x})$$

1

$\rightsquigarrow$ consider $L = L_1 - \lambda L_2$

$$L(\vec{x}, \dot{\vec{x}}) = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} - \lambda(x_1^2 + x_2^2 + x_3^2 - 1)$$

Euler-Lagrange $\rightsquigarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i}$

$$\boxed{\frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}} = -2\lambda x_i} \quad (*)$$

1

Now consider $\vec{l} := \vec{x} \times \vec{p}$ with $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\dot{\vec{x}}}{|\dot{\vec{x}}|}$

$\rightsquigarrow \vec{p} \parallel \dot{\vec{x}}$, or $\underline{\vec{p} \times \dot{\vec{x}} = 0}$

Further, $(*) \rightsquigarrow \dot{\vec{p}} = -2\lambda \vec{x} \rightsquigarrow \underline{\dot{\vec{p}} \times \vec{x} = 0}$

$\rightsquigarrow \dot{\vec{l}} = \dot{\vec{x}} \times \vec{p} + \vec{x} \times \dot{\vec{p}} = 0 \rightsquigarrow \vec{l}$ - const

1

and $\vec{l} \cdot \vec{x} = \vec{x} \cdot (\vec{x} \times \vec{p}) = 0 \rightsquigarrow \boxed{l_1 x_1 + l_2 x_2 + l_3 x_3 = 0}$ (**) with l_i = const

(**) describes a plane that contains the origin
(+) describes a sphere whose center is the origin } $\rightsquigarrow$ <u>The geodesics are great circles.</u>

1