Problem Assignment # 2

01/15/2025
due 01/22/2025

0.2.4. Functional derivative (3 pts)

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)]\right)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

   i) $F = \int dx\, \varphi(x)$

   ii) $F = \int dx\, \varphi^2(x)$

   iii) $F = \int dx\, f(\varphi(x))\, g(\varphi(x))$ where f and g are given functions

   iv) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
   hint: Integrate by parts and assume that the boundary terms vanish.

   v) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

   remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

0.2.5. Massive scalar field (2 pts)

Consider the Lagrangian density for a massive scalar field from the example in ch. 0 §2.5.

a) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi(x))\,(\partial^\mu\phi^*(x)) - \frac{m^2}{2}\,|\phi(x)|^2$$

with ϕ^* the complex conjugate of ϕ. What are the Euler-Lagrange equations now?

b) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

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0.3.1. Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields (6 pts)

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

   hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\,\boldsymbol{E} \times \boldsymbol{B}/\boldsymbol{B}^2$$

   Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

0.3.2. Harmonic oscillator coupled to a magnetic field (4 pts)

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0, charge e). Put the oscillator in a homogeneous time-independent magnetic field $\boldsymbol{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $\boldsymbol{B}$.

0.2.4. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta \varphi(x)} := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \Big(F[\varphi(y) + \epsilon \delta(y-x)] - F[\varphi(y)] \Big)$$

a) Calculate $\delta F / \delta \varphi(x)$ for the following functionals:

   i) $F = \int dx\, \varphi(x)$

   ii) $F = \int dx\, \varphi^2(x)$

   iii) $F = \int dx\, f(\varphi(x))\, g(\varphi(x))$ where f and g are given functions

   iv) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
   hint: Integrate by parts and assume that the boundary terms vanish.

   v) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

   remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is very problematic. It does work for operational purposes, though.

b) Consider a Lagrangian density' $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$ and an action' $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S / \delta \varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}$$

(3 points)

Solution

a) i) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, [\varphi(y) + \epsilon \delta(y-x) - \varphi(y)] = \int dy\, \delta(y-x) = 1$

   ii) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, \Big[(\varphi(y) + \epsilon \delta(y-x))^2 - \varphi(y)^2 \Big] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, \big[2\epsilon \varphi(y) \delta(y-x) + O(\epsilon^2) \big]$
   $= 2\varphi(x)$

   iii) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, [f(\varphi(y) + \epsilon \delta(y-x))] [g(\varphi(y) + \epsilon \delta(y-x)) - f(\varphi(y))\, g(\varphi(y))]$
   $= f'(\varphi(x)) g(\varphi(x)) + f(\varphi(x)) g'(\varphi(x))$ 1pt

   iv) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, \left[\left(\varphi'(y) + \epsilon \frac{d}{dy} \delta(y-x) \right)^2 - (\varphi'(y))^2 \right] = 2 \int dy\, \varphi'(y) \frac{d}{dy} \delta(y-x) = -2\varphi''(x)$

   v) $\frac{\delta F}{\delta \varphi(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, \left[V\left(\varphi'(y) + \epsilon \frac{d}{dy} \delta(y-x) \right) - V(\varphi'(y)) \right]$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int dy\, \left[\epsilon V'(\varphi'(y)) \frac{d}{dy} \delta(y-x) + O(\epsilon^2) \right] = -V''(\varphi'(x))\, \varphi''(x)$ 1pt

b) $0 = \frac{\delta}{\delta \varphi(x)} \int d^4y\, \mathcal{L}(\varphi(y), \partial_\mu \varphi(y))$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y\, [\mathcal{L}(\varphi(y) + \epsilon \delta(y-x), \partial_\mu \varphi(y) + \epsilon \partial_\mu \delta(y-x)) - \mathcal{L}(\varphi(y), \partial_\mu \varphi(y))]$
   $= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y\, \left[\epsilon \delta(y-x) \frac{\partial \mathcal{L}}{\partial \varphi(y)} + \epsilon (\partial_\mu \delta(y-x)) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(y))} + O(\epsilon^2) \right]$
   $= \frac{\partial \mathcal{L}}{\partial \varphi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))}$ EL ✓ 1pt

0.2.5.) a) Treat $\phi(x)$ and $\phi^*(x)$ as independent fields

Minimizing with respect to ϕ^* yields

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = 0$$

and minimizing with respect to ϕ just yields the c.c.

$$(\partial_\mu \partial^\mu + m^2)\,\phi^*(x) = 0$$

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b) Under $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$ we have

$$|\phi(x)|^2 \to |\phi(x)|^2$$

and

$$\partial_\mu \phi(x) \to (\partial_\mu \phi(x))\, e^{i\Lambda(x)} + i(\partial_\mu \Lambda(x))\,\phi(x)\, e^{i\Lambda(x)}$$

$$\partial^\mu \phi^*(x \to (\partial^\mu \phi^*(x))\, e^{-i\Lambda(x)} - i(\partial^\mu \Lambda(x))\,\phi^*(x)\, e^{-i\Lambda(x)}$$

$$\to \partial_\mu \phi(x)\, \partial^\mu \phi^*(x) \to \partial_\mu \phi(x)\, \partial^\mu \phi^*(x) - i\,(\partial^\mu \Lambda(x))(\partial_\mu \phi(x))\,\phi^*(x)$$
$$+ i\,(\partial_\mu \Lambda(x))(\partial^\mu \phi^*(x))\,\phi(x)$$
$$+ (\partial_\mu \Lambda(x))(\partial^\mu \Lambda(x))\,|\phi(x)|^2$$

$$\neq \partial_\mu \phi\, \partial^\mu \phi^*$$

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$\to$ The Lagrangian is not invariant

0.3.1.) a) Eq. of motion: $\quad m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

Let $\vec{B} = (0,0,B)$ and $\vec{E} = (0,E_y,E_z)$

$$\rightarrow \boxed{\begin{array}{l} m\ddot{x} = \frac{e}{c}\dot{y}B \\ m\ddot{y} = eE_y - \frac{e}{c}\dot{x}B \\ m\ddot{z} = eE_z \end{array}} \quad \begin{array}{l}(1)\\(2)\\(3)\end{array}$$

$$(3) \rightarrow \underline{\underline{z(t) = z_0 + v_z^0 t + \frac{eE_z}{2m}t^2}}$$

1

b) Define $\underline{\xi := x + iy}$

$(1) + i\cdot(2) \rightarrow \quad m\ddot{\xi} = ieE_y - i\frac{eB}{c}\dot{\xi}$

Define $\omega := \frac{eB}{mc}$ cyclotron frequency

$$\rightarrow \boxed{\ddot{\xi} + i\omega\dot{\xi} = i\frac{e}{m}E_y} \quad (*)$$

Special solution of inhomogeneous eq.: $\quad \dot{\xi} = \frac{eE_y}{m\omega}$

General solution of homogeneous eq.: $\quad \dot{\xi} = a e^{-i\omega t} \quad (a\in\mathbb{C})$

1 $\quad \rightarrow \underline{\underline{\dot{\xi}(t) = a e^{-i\omega t} + eE_y/m\omega}}$ is the most general solution of $(*)$.

c) With $a = b e^{i\alpha}$, $b,\alpha\in\mathbb{R}$

$\rightarrow \dot{\xi} = b e^{-i(\omega-\alpha)t} + eE_y/(m\omega)$

$\rightarrow \alpha$ just shifts the zero of time $\rightarrow \underline{\underline{\alpha = 0}}$ w.l.o.g.

$\rightsquigarrow \quad \dot{x} + i\dot{y} = b\cos\omega t - ib\sin\omega t + eE_y/m\omega$

$$\rightsquigarrow \quad \boxed{\begin{aligned} \dot{x} &= b\cos\omega t + eE_y/m\omega \\ \dot{y} &= -b\sin\omega t \end{aligned}} \qquad (**)$$

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$\rightsquigarrow \quad \langle \dot{y} \rangle = 0 \,, \quad \langle \dot{x} \rangle = eE_y/m\omega = cE_y/B$ time-averaged velocity

$$= \frac{cE_yB}{B^2} = c(\vec{E}\times\vec{B})_x/B^2$$

in general:

$$\langle \vec{v} \rangle = \frac{c}{B^2}\,\vec{E}\times\vec{B} \qquad \text{drift velocity}$$

condition for $v \ll c$: $\quad E_y/B \ll 1$ — necessary and sufficient condition for non-relativistic approximation

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d) Choose $x(t=0) = 0 = y(t=0)$ w.l.o.g.

$$(**) \rightsquigarrow \quad \boxed{\begin{aligned} x(t) &= \frac{b}{\omega}\sin\omega t + \frac{cE_y}{B}\,t \\ y(t) &= \frac{b}{\omega}(\cos\omega t - 1) \end{aligned}}$$

$\rightsquigarrow$ The path is a trochoid.

To visualize it, put $\omega = 1$ and define $C = cE_y/B$.

$$\rightsquigarrow \quad \boxed{\begin{aligned} x(t) &= b\sin t + Ct \\ y(t) &= b(\cos t - 1) \end{aligned}}$$

This is the projection of the path onto the x-y plane.

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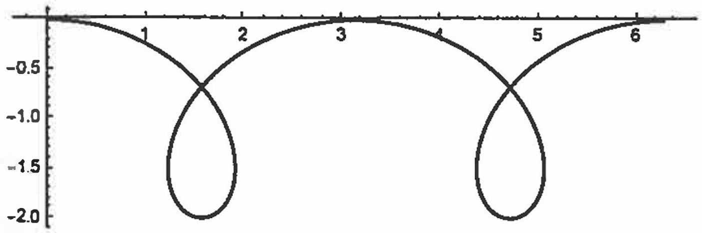
For C < b the trochoid has loops :

```
In[21]:= b = 1;
c = 0.5;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}]
```

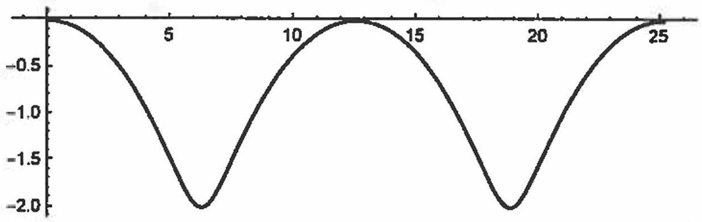
Out[25]=



For C > b it does not :

```
In[86]:= b = 1;
c = 2;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

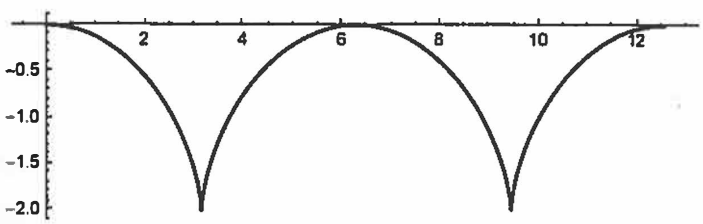
Out[90]=



And for C = b it degenerates into a cycloid :

```
In[91]:= b = 1;
c = 1;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[95]=



1

0.3.2 ·) In addition to the restoring force $-m\omega_0^2 x$, the particle is subject to a Lorentz force

$$\frac{e}{c}\vec{v}\times\vec{B} = \frac{e}{c}(\dot{y}B, -\dot{x}B, 0)$$

$\rightarrow$ The eqs of motion are

$$\left|\begin{array}{ll} \ddot{x} + \omega_0^2 x = \Omega\dot{y} & (1) \\ \ddot{y} + \omega_0^2 y = -\Omega\dot{x} & (2) \\ \ddot{z} + \omega_0^2 z = 0 & (3) \end{array}\right.$$

(1) with $\Omega := eB/mc$ the cyclotron frequency.

(3) $\rightarrow$ For oscillations in the z-direction, the frequency $\omega = \omega_0$ is unchanged

(1)

Define $\xi := x + iy$ and write

(1) $(1) + i\cdot(2) \rightarrow \boxed{\ddot{\xi} + \omega_0^2\xi = -i\Omega\dot{\xi}}$

ansatz: $\xi(t) = \xi_0 e^{i\omega t}$

$$\rightarrow -\omega^2 + \omega_0^2 = \Omega\omega$$

$$\omega^2 + \Omega\omega - \omega_0^2 = 0$$

$$\rightarrow \omega = \frac{1}{2}\left(-\Omega \pm \sqrt{\Omega^2 + 4\omega_0^2}\right) = \pm\sqrt{\omega_0^2 + \Omega^2/4} - \Omega/2$$

$\rightarrow$ The motion in the x-y plane is oscillatory, and eigenfrequencies are

(1) $$\omega_\pm = \sqrt{\omega_0^2 + \Omega^2/4} \pm \Omega/2$$