PHYS 622 Classical Electrodynamics W 2024

Problem Assignment # 3 01/31/2024
due 02/07/2024

0.2.8. **Relativistic motion in parallel electric and magnetic fields**

Consider a relativistic charged particle (mass m, charge e) in parallel homogeneous electric and magnetic fields $\boldsymbol{E} = (0, 0, E)$, $\boldsymbol{B} = (0, 0, B)$.

a) Show that the equation of motion for the z-component of the momentum p_z decouples from p_x and p_y, and that the momentum perpendicular to the z-axis is a constant of motion: $p_x^2 + p_y^2 \equiv p_\perp^2 = \text{const.}$

b) Choose the zero of time such that $p_z(t = 0) = 0$, and show that with a suitable chosen origin the z-component of the particle's position can be written

$$z(t) = \frac{1}{eE}\sqrt{T_0^2 + c^2e^2E^2t^2}$$

where T_0 is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time $t = 0$.

hint: Recall Einstein's law of falling bodies, ch. 0 §3.3.

c) Introduce a parameter φ via $d\varphi/dt = ceB/T(t)$, with $T(t)$ the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$x = \frac{cp_\perp}{eB}\sin\varphi \quad , \quad y = \frac{cp_\perp}{eB}\cos\varphi \quad , \quad z = \frac{T_0}{eE}\cosh(E\varphi/B)$$

and explicitly find the relation between φ and t.

hint: Consider $\pi := p_x + ip_y$ and note that $|\pi| = p_\perp = \text{const.}$ by the result of part a).

d) Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)

1.1.1. **Dual field tensor**

Show that the dual field tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\lambda\kappa}F_{\lambda\kappa}$ obeys $\partial_\mu\tilde{F}^{\mu\nu}(x) = 0$.

hint: First show that $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$, and then relate $\partial_\mu\tilde{F}^{\mu\nu}(x)$ to that expression.

(2 points)

. . . **/over**

1.1.2. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\mathrm{GL}} = \int d\boldsymbol{x} \Big[r \left|\phi(\boldsymbol{x})\right|^2 + c \left|\big[\nabla - iq\boldsymbol{A}(\boldsymbol{x})\big]\phi(\boldsymbol{x})\right|^2 + u \left|\phi(\boldsymbol{x})\right|^4 + \frac{1}{16\pi\mu} F_{ij}(\boldsymbol{x})\, F^{ij}(\boldsymbol{x}) \Big]$$

Here $\boldsymbol{x} \in \mathbb{R}^3$, and $\phi(\boldsymbol{x})$ is a complex-valued field that describes the superconducting matter, $\boldsymbol{A}$ is the Euclidian vector field that comprises the spatial components of the 4-vector $A^\mu = (A^0, \boldsymbol{A})$, and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2, 3)$. μ and q are coupling constants that characterize the vector potential and its coupling to the matter, and r, c and u are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of S_{GL} with respect to all independent fields. (See Problem 0.2.4. You may want to double check against what you get from the Landau-Lifshitz method we used in class.)

b) Show that this theory is invariant under gauge transformations $\phi(x) \to \phi(\boldsymbol{x})\, e^{iq\lambda(\boldsymbol{x})}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\nabla}\lambda(\boldsymbol{x})$.

c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 0.2.5, can be made gauge invariant by coupling $\phi(x)$ to the electromagnetic vector potential $A^\mu(x)$.

   hint: Replace the 4-gradient ∂_μ by $D_\mu = \partial_\mu - iqA_\mu$ and add the Maxwell Lagrangian.

   note: If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)

0.2.8.) a) The Lagrangian is

$$L = L_0 + eEz + \frac{e}{c}\vec{v}\cdot\vec{A}$$

with $L_0 = -mc^2\sqrt{1-v^2/c^2}$

and $\vec{A} = \frac{1}{2}\begin{pmatrix} -By \\ Bx \\ 0 \end{pmatrix} \rightsquigarrow \vec{\nabla}\times\vec{A} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ ✓

$\rightsquigarrow$ $\partial L/\partial z = eE$ independent of B

$\rightsquigarrow$ $\boxed{\dot{p}_z = eE}$ decoupled from p_x, p_y

$\rightsquigarrow$ $p_z(t) = eEt$ (with $t=0$ chosen such that $p_z(t=0)=0$)

1

The forces in x and y-direction are

$$F_x = \frac{e}{c}(\vec{v}\times\vec{B})_x = \frac{eB}{c}v_y \ , \quad F_y = \frac{e}{c}(\vec{v}\times\vec{B})_y = -\frac{eB}{c}v_x$$

$\rightsquigarrow$ $\boxed{\dot{p}_x = \frac{eB}{c}v_y \ , \quad \dot{p}_y = -\frac{eB}{c}v_x}$ (*)

1

where $\vec{p} = (p_x, p_y, p_z) = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$ is the momentum

$\rightsquigarrow$ $\frac{d}{dt}(p_x^2+p_y^2) = 2(p_x\dot{p}_x + p_y\dot{p}_y) = \frac{2meB}{c}\frac{1}{\sqrt{1-v^2/c^2}}(v_xv_y - v_yv_x) = 0$

$\rightsquigarrow$ $p_x^2+p_y^2 =: p_\perp^2 = \text{const}$

1

b) The kinetic energy is

$$T = \vec{v}\frac{\partial L_0}{\partial \vec{v}} - L_0 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \sqrt{m^2c^4+c^2p^2}$$

$$= \sqrt{m^2c^4+c^2p_\perp^2+c^2p_z^2} = \sqrt{T_0^2 + c^2e^2E^2t^2}$$

1

where $T_0 = \sqrt{m^2c^4+c^2p_\perp^2} = T(t=0)$

But $\vec{v} = \frac{\vec{p}}{\sqrt{m^2+p^2/c^2}} = \frac{c\vec{p}}{\sqrt{m^2c^2+p^2}} = \frac{c^2\vec{p}}{T}$ (**)

(1) $\rightsquigarrow \dot{z} = \frac{c^2 p_z}{T} = \frac{c^2 eEt}{\sqrt{T_0^2+c^2e^2E^2t^2}}$

$\rightsquigarrow \underline{z(t)} = z_0 + c^2eE\int_0^t d\tau \frac{\tau}{\sqrt{T_0^2+c^2e^2E^2\tau^2}} = z_0 + \frac{c^2eE}{c^2e^2E^2}\frac{1}{2}\int_0^{ceEt}\frac{dx}{\sqrt{T_0^2+x}}$

$= z_0 + \frac{1}{eE}\frac{1}{2}\int_0^{ceEt/T_0}\frac{dx}{\sqrt{1+x}} = z_0 + \frac{T_0}{eE}\sqrt{1+c^2e^2E^2t^2/T_0^2} - \frac{T_0}{eE}$

$= \frac{1}{eE}\sqrt{T_0^2+c^2e^2E^2t^2}$ with $z_0 = T_0/eE$

(1) cf PHYS 611 Problem 21 ✓

c) Define $\pi = p_x + i p_y$

(*) $\rightsquigarrow \dot{\pi} = \frac{eB}{c}(-i)(v_x+iv_y) \overset{(**)}{=} \frac{eB}{c}(-i)\frac{c^2}{T}\pi = -i\frac{eBc}{T}\pi$

Define φ by $\frac{eBc}{T}dt = d\varphi$

(1) $\rightsquigarrow \frac{d\pi}{d\varphi} = -i\pi \rightsquigarrow \pi = p_\perp e^{-i\varphi}$

$\rightsquigarrow \pi = p_\perp e^{-i\varphi} \overset{(**)}{=} \frac{T}{c^2}(v_x+iv_y) = \frac{T}{c^2}\frac{d}{dt}(x+iy) = \frac{T}{c^2}\frac{eBc}{T}\frac{d}{d\varphi}(x+iy)$

$= \frac{eB}{c}\frac{d}{d\varphi}(x+iy)$

$\rightsquigarrow \frac{dx}{d\varphi} = \frac{cp_\perp}{eB}\cos\varphi, \quad \frac{dy}{d\varphi} = -\frac{cp_\perp}{eB}\sin\varphi$

(1) $\rightsquigarrow \boxed{x = \frac{cp_\perp}{eB}\sin\varphi, \quad y = \frac{cp_\perp}{eB}\cos\varphi}$ with a suitable choice of origin

that $\frac{d\varphi}{dt} = \frac{eBc}{T} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$

$$\rightarrow \quad \varphi = \frac{B}{E} \operatorname{arsinh}\left(\frac{ceEt}{T_0}\right)$$

check: $\frac{d\varphi}{dt} = \frac{B}{E}\frac{ceE}{T_0}\frac{1}{\sqrt{1+\left(\frac{ceEt}{T_0}\right)^2}} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$ ✓

$$\rightarrow \quad \cosh\frac{E\varphi}{B} = \cosh \operatorname{arsinh}\left(\frac{ceEt}{T_0}\right) = \sqrt{1+c^2e^2E^2t^2/T_0^2} = T/T_0$$

while b) $\rightarrow t = T/ceE$

$$\rightarrow \quad t = \frac{T_0}{cE}\cosh\frac{E\varphi}{B}$$

(1)

Now we have the orbit in a parametric representation:

$$x = \frac{cp_\perp}{eB}\sin\varphi, \quad y = \frac{cp_\perp}{eB}\cos\varphi, \quad t = \frac{T_0}{eE}\cosh\frac{E\varphi}{B}$$

where φ is related to t via

$$\varphi = \frac{B}{E}\operatorname{arsinh}\frac{ceEt}{T_0} \quad \text{or} \quad t = \frac{T_0}{ceE}\sinh\frac{E\varphi}{B}$$

(1)

d). The orbit is a helix whose pitch increases exponentially with increasing angle. Choosing the unit of length such that $r_\perp := cp_\perp/eB = 1$ we have

$$x = \sin\varphi, \quad y = \cos\varphi, \quad t = t_0\cosh(E\varphi/B) \quad \text{with } t_0 = \frac{T_0 B}{cp_\perp E}$$

(1)

Here is an example for $t_0 = 1$, $E/B = 0.1$:

p-0.28-4

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In[198]:= z0 = 1
          EB = 0.1
          x[phi_] := Sin[phi]
          y[phi_] := Cos[phi]
          z[phi_] := z0 Cosh[EB phi]
          ParametricPlot3D[{x[phi], y[phi], z[phi]}, {phi, 0, 8 Pi}]
Out[198]= 1

Out[199]= 0.1
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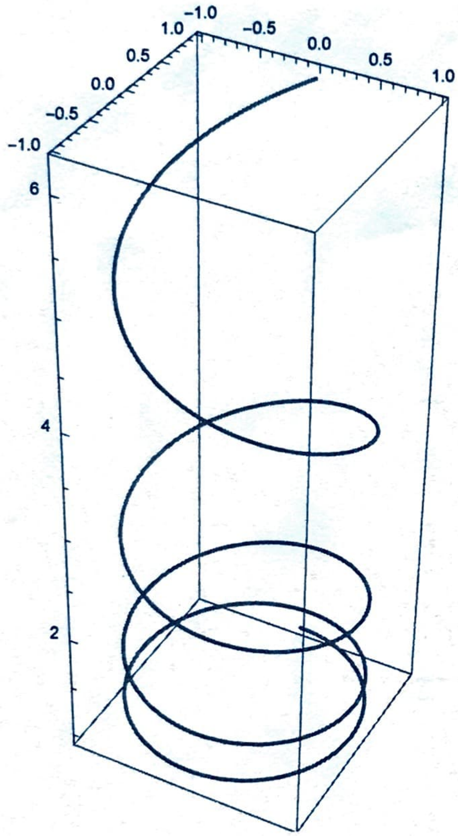
Out[203]=



(1)

For $t \gg \frac{T_0}{ceE}$ we have $T \to ceEt$

$\rightsquigarrow \dot{\varphi} = \frac{ceB}{T} \to \frac{ceB}{ceE}\frac{1}{t} = \frac{B}{E}\frac{1}{t} \to 0$ the angular velocity goes to zero

(1) $z(t) \to ct \rightsquigarrow \dot{z} \to c$ the velocity in z-direction approaches c

For $t \ll \frac{T_0}{ceE}$ we have $\dot{\varphi} = eBc/T_0 + O(t^2)$ const. angular velocity

(1) and $z(t) = z_0 + \frac{1}{2}\frac{c^2 e}{T_0} t^2 + O(t^4)$ Galilean result

1.1.1.) d1 §1.1 def. 1 ⟿ $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\leadsto \underline{\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu}} = \partial^\lambda \partial^\mu A^\nu - \partial^\lambda \partial^\nu A^\mu + \partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu + \partial^\nu \partial^\lambda A^\mu - \partial^\nu \partial^\mu A^\lambda \;\underline{= 0} \quad (*)$$

<u>(i)</u>

$$\leadsto \underline{\partial_\mu \tilde{F}^{\mu\nu}} = \varepsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta}$$

relabel $\stackrel{?}{=} \frac{1}{3} \left[\varepsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta} + \varepsilon^{\alpha\nu\beta\mu} \partial_\alpha F_{\beta\mu} + \varepsilon^{\beta\nu\mu\alpha} \partial_\beta F_{\mu\alpha} \right]$

ε antisymmetric $\stackrel{?}{=} \frac{1}{3} \varepsilon^{\mu\nu\alpha\beta} \left[\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} \right] \stackrel{(*)}{=} \underline{0}$ □

<u>(i)</u>

1.1.2.)

$$S_{GL} = \int d\vec{x} \left[r|\phi(\vec{x})|^2 + c\left|(\vec{\nabla} - ig\vec{A}(\vec{x}))\phi(\vec{x})\right|^2 + u|\phi(\vec{x})|^4 + \frac{1}{16\pi\mu} F_{ij}(\vec{x})F^{ij}(\vec{x}) \right]$$

a) $0 \overset{!}{=} \frac{\delta S_{GL}}{\delta \phi^*(\vec{x})} = r\phi(\vec{x}) + 2u\phi(\vec{x})|\phi(\vec{x})|^2 + c\frac{\delta}{\delta\phi^*(x)}\int d\vec{x}\,(\vec{\nabla}-ig\vec{A})\phi\,(\vec{\nabla}+ig\vec{A})\phi^*$

$= r\phi + 2u\phi|\phi|^2 - c\vec{\nabla}(\vec{\nabla}-ig\vec{A})\phi + ig\vec{A}(\vec{\nabla}-ig\vec{A})\phi$

$= r\phi + 2u\phi|\phi|^2 - c(\vec{\nabla}-ig\vec{A})^2\phi$

(1)

$0 \overset{!}{=} \frac{\delta S_{GL}}{\delta \vec{A}(\vec{x})} = c(-ig)\phi(\vec{\nabla}+ig\vec{A})\phi^* + cig((\vec{\nabla}-ig\vec{A})\phi)\phi^*$

$+ \frac{1}{16\pi\mu}\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\, F_{ij}(\vec{y})F^{ij}(\vec{y})$

(1)

$F_{ij}F^{ij} = (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) = 2\varepsilon^{kij}\varepsilon_{klm}\partial_i A_j \partial^l A^m$

$= 2(\vec{\nabla}\times\vec{A})^2$

$\rightarrow \frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\, F_{ij}F^{ij} = -\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\,\vec{A}(\vec{y})\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{y})))$

$= -2\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x}))$

(1)

$= cig\,\phi^*(\vec{\nabla}-ig\vec{A})\phi + \text{c.c.} - \frac{1}{4\pi\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(x))$

$$\rightarrow \quad -c\left[\vec{\nabla}-ig\vec{A}(\vec{x})\right]^2\phi(\vec{x}) + \left[r + 2u|\phi(\vec{x})|^2\right]\phi(\vec{x}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x})) = 4\pi\mu c i g\,\phi^*(\vec{x})\left[\vec{\nabla}-ig\vec{A}(\vec{x})\right]\phi(\vec{x}) + \text{c.c.}$$

(1)

GL eqs.

b) let $\phi(\bar{x}) \to \phi(\bar{x})\, e^{ig\lambda(\bar{x})}$, $\vec{A}(\bar{x}) \to \vec{A}(\bar{x}) + \vec{\nabla}\lambda(\bar{x})$

$\to$ $|\phi(\bar{x})|^2 \to |\phi(\bar{x})|^2$

and $F_{ij}(\bar{x}) = \partial_i A_j - \partial_j A_i \to \partial_i (A_j + \partial_j \lambda) - \partial_j (A_i + \partial_i \lambda) = F_{ij}(\bar{x})$

(1) Finally,

$(\vec{\nabla} - ig\vec{A})\phi \to (\vec{\nabla} - ig\vec{A} - ig\vec{\nabla}\lambda)\phi e^{ig\lambda} =$

$= (\vec{\nabla}\phi) e^{ig\lambda} + ig(\vec{\nabla}\lambda)\phi e^{ig\lambda} - ig\vec{A}\phi e^{ig\lambda} - ig(\vec{\nabla}\lambda)\phi e^{ig\lambda}$

$= e^{ig\lambda}(\vec{\nabla} - ig\vec{A})\phi$

$\to |(\vec{\nabla} - ig\vec{A})\phi|^2 \to |(\vec{\nabla} - ig\vec{A})\phi|^2$

(1) $\to$ $\mathcal{F}_{GL}$ is gauge invariant

c) Modify $\mathcal{L}$ from Problem 3b) to read

$$\mathcal{L} = (D_\mu \phi(x))(D^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x) \quad (*)$$

where $D_\mu = \partial_\mu - igA_\mu$

let $\phi(x) \to \phi(x)\, e^{ig\lambda(x)}$, $A^\mu(x) \to A^\mu(x) + \partial^\mu \lambda(x)$

$\to |\phi|^2 \to |\phi|^2$ and $F_{\mu\nu}F^{\mu\nu} \to F_{\mu\nu}F^{\mu\nu}$

$D_\mu \phi \to (\partial_\mu - igA_\mu - ig\partial_\mu \lambda)\phi e^{ig\lambda}$

$= e^{ig\lambda}(\partial_\mu + ig\partial_\mu \lambda - igA_\mu - ig\partial_\mu \lambda)\phi = e^{ig\lambda} D_\mu \phi$

$\to (D_\mu \phi)(D^\mu \phi)^* \to (D_\mu \phi)(D^\mu \phi^*)$

(1) $\to$ (*) is gauge invariant