PHYS 610 Mathematical Methods for Scientists F 2025

Final Exam 12/05/2025 - 12/12/2025

This is a limited-time open-book exam. You can use any inanimate resources you like, but please don't get help from live ones.

You can choose any contiguous twentyfour-hour period you like for working on the exam, but once you open it you must finish it within the next twentyfour hours. When you are done, scan it and email it to dbelitz@you-know-where. Please try to keep the scan under 5MB. The exam is due no later than Friday, Dec. 12, 5pm PST. Since different people will be working on the exam at different times I will not answer any questions once the exam has been published (unless it's a true emergency). If you feel there is something wrong or ambiguous about the formulation of a part of the problem, clearly state your assumptions and keep going.

Fermionic Matsubara frequency sum [21 pts: Part a) 8 pts, Part b) 6 pts, Part c) 7 pts]

a) Let $g(z)$ have isolated poles (**not necessarily simple ones!**) at z_j ($j = 1, 2, \ldots$) and be otherwise holomorphic. Let g to to zero faster than $1/|z|$ for $|z| \to \infty$. Consider the infinite sum

$$S = T \sum_{n=-\infty}^{\infty} g(i\omega_n)$$

with $\omega_n = 2\pi T(n + 1/2)$ ("fermionic Matsubara frequency") and $T > 0$ the temperature. Show that

$$S = \sum_j \text{Res}\Big|_{z=z_j} \left(f(z)\, g(z)\right)$$

where

$$f(z) = \frac{1}{e^{z/T} + 1}$$

is the Fermi-Dirac function. Show that this reduces to a straightforward analog of the result of Problem II.3.3 if the poles of g are simple.

b) Let $x \in I\!R$. Show that

$$\lim_{T \to 0} f(x) = \Theta(-x)$$

in a distribution-limit sense, with Θ the Heaviside step function. What does this imply for the derivative $f'(x) = df/dx$ at $T = 0$?

Note: If you have trouble proving this, assume it's true and use it to do part c).

c) Consider

$$g_\xi(z) = \frac{1}{(z - \xi)^2} \frac{1}{(z + i\Omega_m - \xi)^2}$$

with $\Omega_m = 2\pi T m$ a bosonic Matsubara frequency and $\xi \in I\!R$. Calculate

$$J = \int d\xi\; T \sum_n g_\xi(i\omega_n)$$

in the limit $T \to 0$.

hint: What are $f(\xi - i\Omega_m)$ and $f'(\xi - i\Omega_m)$?