I.1.4 Parabolic Mapping (4 pts)

Consider $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = a\,n^2 + b\,n + c$, with $a, b, c \in \mathbb{Z}$.

a) For which triplets (a, b, c) is f surjective?

b) For which (a, b, c) is f injective?

Solution

a) $f(n)$ has a global extremum if $a \neq 0$. $\Rightarrow a = 0$ is necessary for f to be surjective.
Now consider $f(n) = bn + c$.
If $b = 0$, then $f(n) \equiv c$ and hence f is not surjective.
If $b \geq 2$ or $b \leq -2$, then $f(n)$ never equals $c \pm 1$, and hence f is not surjective.
If $b = \pm 1$ with arbitrary c, then $f(n)$ covers all of $\mathbb{Z}$.

$\Rightarrow f$ is surjective for $(a, b, c) = (0, \pm 1, c \in \mathbb{Z})$. 2pts

b) For f to be injective, $f(n) = f(m)$ must imply $n = m$.
Let $n = m + x$ with $x \in \mathbb{Z}$. Then $f(n) = f(m)$ takes the form

$$am^2 + bm = am^2 + 2axm + ax^2 + bm + bx$$

$$\Rightarrow \; ax^2 + (2am + b)x = 0 \qquad (*)$$

$x = 0$ is always a solution of (*), which implies $n = m$.
For $x \neq 0$ the only solution of (*) is

$$x = -2m - b/a$$

As long as this solution is $\notin \mathbb{Z}$, f is injective.
$\Rightarrow$ For f to be injective, b must not be divisible by a in $\mathbb{Z}$.

$\Rightarrow f$ is injective for $(a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus a\mathbb{Z},\ c \in \mathbb{Z})$. 2pts