I.1.5 Equivalence relations

Consider a relation $\sim$ on a set X as in ch. 1 §1.3 def. 1, but with the properties

i) $x \sim x \quad \forall x \in X$ (reflexivity)

ii) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$ (symmetry)

iii) $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$ (transitivity)

Such a relation is called an *equivalence relation.* Which of the following are equivalence relations?

a) n divides m on $\mathbb{N}$.

b) $x \leq y$ on $\mathbb{R}$.

c) g is perpendicular to h on the set of straight lines $\{g, h, \ldots\}$ in the cartesian plane.

d) a equals b modulo n on $\mathbb{Z}$, with $n \in \mathbb{N}$ fixed.
hint: "a equals b modulo n", or $a = b \bmod(n)$, with $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$, is defined to be true if $a - b$ is divisible on $\mathbb{Z}$ by n; i.e., if $(a - b)/n \in \mathbb{Z}$.

(3 points)

Solution

a) No, since it is not symmetric. E.g., $2 \sim 4$, but $4 \nsim 2$.

b) No, since it is not symmetric. E.g., $2 \leq 4$ but $4 \nleq 2$.

c) No, since it is not reflexive: No line is perpendicular to itself. 1pt

d) Yes.

Proof. i) $a - a = 0$ is divisible by $n \quad \Rightarrow \quad a = a \bmod(n)$

ii) Let $a = b \bmod(n) \Rightarrow \exists\, k \in \mathbb{Z} : a - b = n$
$\Rightarrow b - a = (-k)n \Rightarrow b - a$ is divisible by n
$\Rightarrow b = a \bmod(n)$

iii) Let $a = b \bmod(n)$ and $b = c \bmod(n)$
$\Rightarrow \exists\, k, \ell \in \mathbb{Z} : a - b = kn$ and $b - c = \ell n$
$\Rightarrow a - c = (a - b) + (b - c) = kn + \ell n = (k + \ell)n$ with $k + \ell \in \mathbb{Z}$
$\Rightarrow a = c \bmod(n)$

$\Rightarrow a = b \bmod(n)$ is an equivalence relation on $\mathbb{Z}$. $\square$

2pts