I.1.6 Bounds for $n!$

Prove by mathematical induction that

$$n^n/3^n < n! < n^n/2^n \quad \forall\ n \geq 6$$

hint: $(1 + 1/n)^n$ is a monotonically increasing function of n that approaches Euler's number e for $n \to \infty$.

(4 points)

Solution

Proof. First prove $n^n/3^n < n!\ \forall n \geq 6$:
For $n = 6$ we have $6^6/3^6 = 2^6 = 64 < 720 = 6!$, so the inequality holds.
Now assume $m^m/3^m < m!$. Then it follows that

$$\begin{aligned}
\frac{(m+1)^{m+1}}{3^{m+1}} &= \frac{m^m}{3^m}\,\frac{1}{3}\,(1+1/m)^m(m+1) \\
&< \frac{m^m}{3^m}\,\frac{e}{3}\,(m+1) \qquad \text{by the hint} \\
&< \frac{m^m}{3^m}\,(m+1) < m!\,(m+1) \qquad \text{by the assumption} \\
&= (m+1)!
\end{aligned}$$

2pts

Now prove $n^n/2^n > n!\ \forall n \geq 6$:
For $n = 6$ we have $6^6/2^6 = 3^6 = 729 > 720 = 6!$, so the inequality holds.
Now assume $m^m/2^m > m!$. Then it follows that

$$\begin{aligned}
\frac{(m+1)^{m+1}}{2^{m+1}} &= \frac{m^m}{2^m}\,\frac{(1+1/m)^m}{2}(m+1) \\
&\geq \frac{m^m}{2^m}\,(m+1) \qquad \text{by the hint} \\
&> m!(m+1) \qquad \text{by the assumption} \\
&= (m+1)!
\end{aligned}$$

□

2pts